

A Short Solution of The Riemann Hypothesis Problem

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ABSTRACT. In 1859, Georg Friedrich Bernhard Riemann had announced the following conjecture, called the Riemann hypothesis : *The nontrivial roots (zeros) $s = \sigma + it$ of the zeta function, defined by:*

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) > 1$$

have real part $\sigma = \frac{1}{2}$. In this paper, it was used the expression of $\zeta(1 - \bar{s})$ in function of $\zeta(s)$, it implies to the proof that $\Re(s) = \sigma = \frac{1}{2}$.

RÉSUMÉ: En 1859, Georg Friedrich Bernhard Riemann a énoncé la conjecture suivante, appelée "Hypothèse de Riemann" : *les zéros non triviaux $s = \sigma + it$ de la fonction ζ , définie par :*

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ avec } \Re(s) > 1$$

ont une partie réelle $\sigma = \frac{1}{2}$. Dans cet article, on a utilisé l'expression de $\zeta(1 - \bar{s})$ en fonction de $\zeta(s)$, ce qui conduit à démontrer que $\Re(s) = \sigma = \frac{1}{2}$.

Keywords: Zeta function, Riemann hypothesis, complex analysis.

This paper is dedicated to the memory of my father who taught me arithmetic.

*To my wife, my daughter, my son and my granddaughter Rayhane
To the memory of the Prof. Mohamed AMARA, former Dean of the
Faculty of Sciences of Tunis*

1. INTRODUCTION

In 1859, G.F.B. Riemann had announced the following conjecture [1] known Riemann hypothesis:

Conjecture 1.1. Let $\zeta(s)$ be the complex function of the complex variable $s = \sigma + it$ defined by the analytic continuation of the function:

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) = \sigma > 1$$

over the whole complex plane, with the exception of $s = 1$. Then the nontrivial zeros of $\zeta(s) = 0$ are written as :

$$s = \frac{1}{2} + it$$

In this paper, our idea is to start with the Riemann's ζ function where we do not need to manipulate any expression of it in the critical band $0 < \Re(s) < 1$. We arrive to give the proof that $\sigma = \frac{1}{2}$.

1.1. The function zeta(s). We denote $s = \sigma + it$ the complex variable of $\mathbb{C}$. For $\Re(s) = \sigma > 1$, let ζ_1 be the function defined by :

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) = \sigma > 1$$

We know that with the previous definition, the function ζ_1 is an analytical function of s . Denote by $\zeta(s)$ the function obtained by the analytic continuation of $\zeta_1(s)$ to the whole complex plane, minus the point $s = 1$, then we recall the following theorem [2]:

Theorem 1.2. The function $\zeta(s)$ satisfies the following :

1. $\zeta(s)$ has no zero for $\Re(s) > 1$;
2. the only pole of $\zeta(s)$ is at $s = 1$; it has residue 1 and is simple;
3. $\zeta(s)$ has trivial zeros at $s = -2, -4, \dots$;
4. the nontrivial zeros lie inside the region $0 \leq \Re(s) \leq 1$ (called the critical strip) and are symmetric about both the vertical line $\Re(s) = \frac{1}{2}$ and the real axis $\Im(s) = 0$.

The vertical line $\Re(s) = \frac{1}{2}$ is called the critical line.

We suppose the case $t = 0$, but we known the following proposition:

Proposition 1.1. For all $s = \sigma$ real with $0 < \sigma < 1$, $\zeta(s) < 0$ and $\eta(s) > 0$.

$\eta(s)$ is the Dirichlet function. We consider in the following $s = \sigma + i.t$ with $0 < \sigma < 1$ and $t > 0$. We have also the theorem (see page 16, [3]):

Theorem 1.3. For all $t \in \mathbb{R}$, $\zeta(1 + it) \neq 0$.

So, we take the critical strip as the region defined as $0 < \Re(s) < 1$.

2. THE PROOF THAT THE ZEROS OF THE FUNCTION $\zeta(s)$ ARE ON THE CRITICAL LINE

Proof. We consider the complex number $s = \sigma + it, t \neq 0$ with $0 < \sigma < 1/2$ and its symmetric about the vertical line $\Re(s) = \frac{1}{2}$, $s' = 1 - \bar{s} = 1 - (\sigma - it) = 1 - \sigma + it = \sigma' + it'$ with $t' = t > 0$, $1/2 < \sigma' < 1$. The complex function ζ is an analytical function of s on the strip $]0, 1[$, we can write:

$$(2.1) \quad \zeta(s') = \zeta(s) + (s' - s) \cdot \zeta'(s) + \frac{1}{2}(s' - s)^2 \cdot \zeta''(s) + \frac{1}{6}(s' - s)^3 \cdot \zeta'''(s) + \dots$$

The main paragraph is the following: we consider that $s = \sigma + it$ is a zero of the function zeta with $0 < \sigma < 1/2$. It follows from the theorem (1.2) $s' = \sigma' + it'$ is also a zero of the function zeta, then $\zeta(s) = 0 = \zeta(s')$. The equation (2.1) becomes:

$$(2.2) \quad \begin{cases} 0 = 0 + (s' - s) \cdot \left[\zeta'(s) + \frac{1}{2}(s' - s) \cdot \zeta''(s) + \frac{1}{6}(s' - s)^2 \cdot \zeta'''(s) + \dots \right] \implies \\ \left\{ \begin{array}{l} s' - s = 0 \implies 1 - \sigma + it - (\sigma + it) = 1 - 2\sigma = 0 \implies \boxed{\sigma = \frac{1}{2}} \\ \text{or, } \zeta'(s) + \frac{1}{2}(s' - s) \cdot \zeta''(s) + \frac{1}{6}(s' - s)^2 \cdot \zeta'''(s) + \dots = 0 \implies \zeta'(s) = 0 \end{array} \right. \end{cases}$$

As $\sigma = \Re(s) = \frac{1}{2}$, then $\zeta(s) = \zeta(\frac{1}{2} + it) = 0 \implies \zeta'(\frac{1}{2} + it) = \zeta'(s) = 0$. The proof of the Riemann Hypothesis is finished.

Q.E.D.

□

We therefore announce the important theorem as follows:

Theorem 2.4. *The Riemann hypothesis is true:*

All nontrivial zeros of the function $\zeta(s)$ with $s = \sigma + it$ lie on the vertical line $\Re(s) = \frac{1}{2}$.

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