

A Recreational Exploration of Near-Miss Constructions for a 3×3 Magic Square of Square Numbers

The Error Function, Eisenstein Triangles, and Continued Fractions as Tools for Investigating Progressively Better Approximations

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Abstract

This work falls within the scope of recreational mathematics and the experimental exploration of the order-3 magic square of squares. Instead of resolving the open problem of the existence of such an object, the research focuses on “near-miss” structures. The analysis is based on the reduction of the original problem to three three-term arithmetic progressions of square numbers with a common difference r , whose first terms also form an arithmetic progression.

By relaxing the condition requiring the initial terms to form an arithmetic progression, this paper introduces a percentage error function p . This function allows for a quantitative assessment of the discrepancy between the differences of these terms, and consequently, a quantitative measure of how closely a configuration approximates a fully magic structure.

Subsequently, an algorithm developed for this study was applied to generate all three-term progressions of integer squares with a common difference r . To optimize calculations and eliminate redundant operations, a filter based on the theory of quadratic residues in the ring $\mathbb{Z}_{240}$ was implemented. Ultimately, a deterministic exploration of the differences r from 2 to 70,000,000,000,000 was conducted.

Next, a geometric parametrization utilizing 120-degree Eisenstein triangles is presented. This enabled a dimensionality reduction of the error optimization problem to a function of a single real variable. It was demonstrated that the theoretical optimum point with zero error corresponds to an irrational number, preventing its direct realization in integers. It was proven, however, that despite structural limitations, it is possible to construct a parametric matrix generating semi-magic squares with equal sums in rows, columns, and on one diagonal.

Diophantine approximation and continued fractions yielded an infinite sequence of rational approximations where the error function converges asymptotically to zero, while the ratio of the diagonal sums remains close to 1. This is illustrated by a numerical example where the ratio yields a value on the order of $1 - 7,3506 \cdot 10^{-103230}$.

The results offer a new method for the assessment of near-miss structures based on the error function p . They outline an approach based on Eisenstein triangles,

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for generating configurations where the error p tends to zero and the diagonal sum ratio approaches 1. This work does not prove the non-existence of the magic square of squares, but provides a universal research framework for approaching ideal structures.

Keywords: magic square of squares, semimagic square of squares, Eisenstein triangles, Diophantine approximation, continued fractions.

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1 Introduction

A magic square of order n , for $n \in \mathbb{N} \setminus \{0\}$, is a square array (matrix) of dimensions $n \times n$ whose cells are filled with numbers in such a way that the sum of the entries in every row, every column, and both diagonals is the same (the so-called magic constant) [1].

The history of magic squares dates back to antiquity [2]. Over the following centuries, these objects attracted interest and found applications in, among other fields, art [3], astrology [4], and occult traditions [5], while their properties and construction methods became the subject of numerous studies. One outcome of research into such structures is the content of a letter sent by Leonhard Euler to Joseph-Louis Lagrange in 1770, in which Euler presented in detail a magic square of order 4 composed of square numbers [6]. In the late nineteenth century, in 1890–1891, Georges Pfeffermann developed this idea further by constructing the first known bimagic squares of orders 8 and 9. A bimagic square is a magic square that retains its magic property when each of its entries is replaced by its square [7, 8].

Contemporary interest in magic squares composed of square numbers has focused largely on the question of whether there exists a magic square of order 3 in which all nine entries are perfect squares. This problem was formulated by Martin LaBar in 1984 [9] and subsequently included by Richard Guy in his book *Unsolved Problems in Number Theory*, published in 1994 [10]. It gained wider recognition in 1996 through two articles by Martin Gardner published in *Quantum* magazine [11, 12], which offered a prize of \$100 for a solution to the problem. Although, during the first decade of the twenty-first century, Christian Boyer successfully constructed magic squares of square numbers of orders 5, 6, and 7, and squares of even higher orders are now also known, the puzzle of the smaller, order-3 square remains unsolved. To date, neither an example of such a square nor a proof demonstrating that its existence is impossible has been presented [13].

In the absence of a resolution to the problem of a magic square of square numbers of order 3, research has also been conducted into constructions that partially satisfy its requirements (so-called *near-miss* constructions). Two main approaches to this type of investigation can be distinguished:

- **The first approach** preserves the magic property in full, requiring all eight sums—corresponding to the three rows, three columns, and two diagonals—to be equal. At the same time, the condition concerning the type of entries is relaxed: the objective is to maximize the number of distinct perfect squares among the nine entries of the array. In this category, the record is held by a construction published by Andrew Bremner in 1999, containing seven distinct perfect squares while maintaining equality among all eight sums. Notably, to date, no other construction containing seven distinct perfect squares has been found that is not equivalent to Bremner’s square or a scalar multiple of it.
- **The second approach** reverses these priorities. In this case, all nine entries must be perfect squares, while the requirement that all sums be equal is abandoned. For such an array, the number of the eight sums that share the same value is examined. The objective is therefore to maximize the number of equal sums while preserving the condition that every entry is a perfect square. The best-known constructions in this category achieve seven equal sums out of the eight possible sums [14].

In the remainder of this paper, the problem will first be reduced to an algebraic form by expressing a magic square in terms of three three-term arithmetic progressions of square numbers and employing Euclid's parametrization of Pythagorean triples (primitive solutions). The primitive structures obtained in this way will then be scaled by iterating over a square multiplier, while modular arithmetic techniques will be applied to reduce the search space. A key aspect of the adopted methodology is the possibility of partitioning the global task into disjoint ranges (subproblems) that are entirely separate from one another, allowing them to be processed independently and without conflicts.

Next, a percentage error function p will be introduced to quantify how closely a given construction approaches the condition of being fully magic.

In the subsequent stage, a parametrization using Eisenstein triangles with an angle of 120° will be employed to unify the differences of the progressions under consideration and reduce the problem to a function of a single variable. On this basis, a theoretical optimum point will be determined, and its algebraic properties, particularly its irrationality, will then be investigated. Finally, using Diophantine approximation and continued fractions, it will be demonstrated how to construct a sequence of *near-miss* structures for which the percentage error asymptotically approaches zero, while the ratio of the diagonal sums remains close to 1.

2 Algebraic Reduction of the Problem

2.1 Theoretical Foundations of the Problem Reduction

It has been shown that the problem of determining whether there exists a magic square of order 3 composed exclusively of square numbers can be reduced to the problem of finding three three-term arithmetic progressions of square numbers with a common difference (r), whose first terms form an arithmetic progression with a common difference (d_0) [10].

The numbers appearing in such a square can therefore be expressed in the form:

$$\begin{aligned} &A, \quad A + r, \quad A + 2r, \\ &A + d_0, \quad A + d_0 + r, \quad A + d_0 + 2r, \\ &A + 2d_0, \quad A + 2d_0 + r, \quad A + 2d_0 + 2r, \end{aligned}$$

where each of the nine terms must be a perfect square. The corresponding arrangement of these elements satisfying the magic property can be represented by the matrix:

$$\begin{pmatrix} A + r & A + 2r + 2d_0 & A + d_0 \\ A + 2d_0 & A + d_0 + r & A + 2r \\ A + d_0 + 2r & A & A + 2d_0 + r \end{pmatrix}.$$

In each of the eight equivalent configurations obtained by applying the symmetries of the square, the magic constant takes the form:

$$S = 3(A + r + d_0).$$

The algebraic structure of the matrix under consideration exhibits complete symmetry with respect to the step parameters r and d_0 . Interchanging the roles of these variables ($r \leftrightarrow d_0$) corresponds to a geometric transformation of the matrix involving transposition combined with reflection, which fully preserves its magic properties.

A consequence of this duality is the structural closure of the solution space: if a pair (r, d_0) defines a valid system of arithmetic progressions forming a magic square, then the pair (d_0, r) also constitutes a valid solution, generating an isomorphic configuration with the opposite spatial orientation.

The search for a magic square satisfying the conditions under consideration therefore reduces to finding three three-term arithmetic progressions of square numbers with a common difference r , whose first terms are related by an arithmetic progression with common difference d_0 . Owing to the symmetry described above, every solution obtained in the form of a pair (r, d_0) automatically induces a dual solution corresponding to the pair (d_0, r) .

2.2 Modular Filtering of the Search Space Based on the Theory of Quadratic Residues

To narrow the search space and eliminate algebraically inconsistent values, an initial modular filter based on the theory of quadratic residues was introduced.

The modulo M operation maps the infinite set of integers $\mathbb{Z}$ onto the finite ring of residue classes $\mathbb{Z}_M = \{0, 1, 2, \dots, M-1\}$. The elements of this ring are equivalence classes under the congruence relation modulo M , meaning that each integer is identified with the remainder obtained when it is divided by M .

An integer $q \in \mathbb{Z}_M$ is called a quadratic residue modulo M if there exists an $x \in \mathbb{Z}_M$ such that the congruence $x^2 \equiv q \pmod{M}$ holds. Otherwise, q is called a quadratic nonresidue. Since the square of any integer can take, modulo M , only values belonging to the set of quadratic residues, every entry of the sought magic square must satisfy this property.

By examining the behavior of the matrix of generator structures in the ring $\mathbb{Z}_M$, all triples of potential residues $(A_{\text{res}}, r_{\text{res}}, d_{\text{res}})$ corresponding to the starting value A , the internal difference r , and the difference d_0 are analyzed. This approach is used to define admissible residue classes (congruence classes) that satisfy the necessary conditions for generating a valid solution. Referring to the general matrix structure presented in Section 2.1, only those parameter residue classes for which all nine entries of the grid yield quadratic residues are retained for further analysis.

Testing moduli M in the range from 2 to 250 showed that the modulo 240 system provides the most effective filter within this range. It allows 97.5% of inadmissible residue classes to be immediately eliminated for both r and d_0 . Let $\mathcal{R}$ denote the set of admissible residues modulo 240:

$$\mathcal{R} = \{0, 24, 96, 120, 144, 216\}.$$

Consequently, the parameters r and d_0 are restricted exclusively to congruence classes satisfying the conditions:

$$r \bmod 240 \in \mathcal{R}, \quad d_0 \bmod 240 \in \mathcal{R}.$$

2.3 The Percentage Error Function p

In the classical formulation, finding an exact magic square of square numbers requires two conditions to be satisfied simultaneously: the three arithmetic progressions must have identical common differences (denoted by r_1, r_2, r_3), and the differences between the first terms of these progressions must also be identical. Let d_1 denote the difference between

the first terms of the second and first progressions, and let d_2 denote the difference between the first terms of the third and second progressions.

In an ideal magic square, the identities $r_1 = r_2 = r_3 = r$ and $d_1 = d_2 = d_0$ hold. Since no nontrivial solution to this problem has been presented in the literature to date, a natural research approach is to relax one of these conditions and select a specific optimization strategy.

Owing to the dual symmetry established in Section 2.1, the choice of optimization direction is mathematically equivalent. The construction allows for either the search for progressions with identical common differences ($r_1 = r_2 = r_3 = r$), while minimizing the deviation between d_1 and d_2 , or the reverse approach, in which $d_1 = d_2 = d_0$ is fixed while the deviations among the common differences r_i , for $i \in \{1, 2, 3\}$, are minimized. Without loss of generality, the first approach is adopted in this paper: the common difference is fixed at r , while the matching criterion is based on analyzing the discrepancy between d_1 and d_2 . It is worth noting that, owing to the duality of the parameters, the target value d_0 may alternatively serve as the sought-after common difference r .

To establish consistent notation, we assume that the three arithmetic progressions of square numbers under consideration are always ordered according to their first terms. Denoting these first terms by A_1, A_2, A_3 , we adopt the convention:

$$A_1 < A_2 < A_3.$$

Consequently, the differences $d_1 = A_2 - A_1$ and $d_2 = A_3 - A_2$ are positive by definition. This eliminates ambiguities arising from permutations of the progressions and ensures consistency in subsequent analyses.

To measure how closely a given numerical configuration approaches a magic structure, we define the percentage error function p (also known as the *percentage difference*), which is used to compare two values of equal significance, neither of which is treated as a reference value:

$$p = \frac{|d_1 - d_2|}{\frac{d_1 + d_2}{2}} \cdot 100\%$$

Assuming that $d_1, d_2 \in \mathbb{N}_+$, the range of values of the percentage error function is restricted to $p \in [0\%, 200\%]$. In an ideal magic square, this function takes the value $p = 0\%$.

Within the adopted methodology, the differences between the first terms of the three arithmetic progressions are analyzed, each of which, by assumption, consists exclusively of square numbers. Since the initial structure is defined in terms of progressions with precisely these properties, the condition that all nine entries of the potential magic square are perfect squares is satisfied by construction.

Consequently, the sole remaining optimization criterion is the minimization of the percentage error p . Introducing this measure provides an evaluation scale that makes it possible to quantify how closely a given configuration approaches an ideal magic square. Since a smaller value of p directly reflects a reduction in the discrepancy between d_1 and d_2 , this process establishes a well-defined geometric path toward the complete construction of the sought-after matrix. An open and central research question therefore arises: how close can we get to the theoretical optimum $p = 0\%$ within a restricted numerical search space?

2.4 A Deterministic Algorithm for Constructing Arithmetic Progressions of Square Numbers

2.4.1 Problem Formulation and Algebraic Parametrization

This section presents a method for generating all three-term arithmetic progressions of square numbers (x^2, y^2, z^2) with a prescribed common difference $r \in [R_{\min}, R_{\max}]$. The strategy is based on three mutually complementary components: an isomorphic reduction to Pythagorean triples, Euclid's parametrization, and a parametrization of the progression through residue classes using the congruence filter described in Section 2.2. This filter is applied in two stages: first, at the level of selecting values of r from the interval $[R_{\min}, R_{\max}]$, by eliminating in advance all values of r for which $r \bmod 240 \notin \mathcal{R}$; and second, at the level of enumerating the scaling parameter k for each fixed primitive triple (m, n) .

Problem Formulation and Algebraic Parametrization The condition for the existence of an arithmetic progression (x^2, y^2, z^2) with common difference r takes the form:

$$y^2 - x^2 = z^2 - y^2 = r,$$

which, after rearrangement, is equivalent to the equation:

$$x^2 + z^2 = 2y^2. \quad (\text{E1})$$

Reduction to Pythagorean Triples We reduce equation (E1) by introducing the variables:

$$a = \frac{z - x}{2}, \quad b = \frac{z + x}{2},$$

so that $x = b - a$ and $z = b + a$. Substituting these expressions into (E1), we obtain:

$$(b - a)^2 + (b + a)^2 = 2y^2 \iff a^2 + b^2 = y^2. \quad (\text{E2})$$

Equation (E2) is the classical Pythagorean equation. Every solution (a, b, y) of this equation generates a solution (x^2, y^2, z^2) to the original problem. Importantly, the common difference r of the progression can be expressed in the new variables as:

$$r = y^2 - x^2 = (a^2 + b^2) - (b - a)^2 = 2ab. \quad (\text{E3})$$

Equation (E3) establishes an isomorphism: the common difference of an arithmetic progression of square numbers is twice the product of the legs of a Pythagorean triangle.

Euclid's Parametrization Every primitive Pythagorean triple (A, B, C) , that is, one satisfying $\gcd(A, B, C) = 1$, is given by Euclid's formulas [15]:

$$A = m^2 - n^2, \quad B = 2mn, \quad C = m^2 + n^2,$$

where $m > n \geq 1$, $\gcd(m, n) = 1$, and $m \not\equiv n \pmod{2}$. A general Pythagorean triple is obtained by scaling:

$$(a, b, y) = (kA, kB, kC), \quad k \in \mathbb{N}_+.$$

Substituting into (E3), we obtain the following formula for subsequent analysis:

$$r = 2ab = 2k^2 AB = 4k^2 mn(m^2 - n^2). \quad (\text{E4})$$

Formula (E4) provides a complete parametrization of the set of admissible values of r in terms of the triple (m, n, k) .

Parametrization of the Progression Through Residue Classes Observe that, for a fixed primitive triple (m, n) , the common difference r as a function of the scaling parameter k is quadratic:

$$r(k) = r_0 \cdot k^2, \quad \text{where } r_0 = 2AB = 4mn(m^2 - n^2).$$

The remainder of $r(k)$ upon division by 240 depends only on the remainder $\rho_0 = r_0 \bmod 240$ and the remainder $\kappa = k \bmod 240$. According to the modular filtering procedure described in Section 2.2, for a generated progression to qualify as a candidate solution, its common difference r must satisfy $r \equiv \rho \pmod{240}$ for some $\rho \in \mathcal{R}$, where $\mathcal{R} = \{0, 24, 96, 120, 144, 216\}$ is the set of admissible residues defined in Section 2.2.

Accordingly, for each fixed value of ρ_0 , we determine in advance the set of admissible scaling residues κ :

$$\mathcal{K}(\rho_0) = \{\kappa \in \{1, \dots, 240\} : \rho_0 \cdot \kappa^2 \equiv \rho \pmod{240} \text{ for some } \rho \in \mathcal{R}\}.$$

This preliminary tabulation makes it possible to replace a linear scan over $k = 1, 2, 3, \dots$ with a stepped parametrization:

$$k = \kappa + 240t, \quad t \in \mathbb{N}_0,$$

where κ takes only the previously determined admissible residue values from the set $\mathcal{K}(\rho_0)$. In this way, we enumerate only those values of k that guarantee that the generated r satisfies $r \bmod 240 \in \mathcal{R}$, excluding rejected congruence classes in advance.

Bounding the Parameter Ranges From the inequality

$$r = 4k^2mn(m^2 - n^2) \leq R_{\max},$$

under the assumptions $k \geq 1$ and $n \geq 1$, it follows that, for a fixed m , the minimum value of r is

$$r_{\min}(m) = 4m(m^2 - 1),$$

attained when $n = 1$ and $k = 1$. Consequently, the maximum value of the parameter m is determined exactly as the largest integer satisfying

$$4m(m^2 - 1) \leq R_{\max}.$$

For each pair (m, n) satisfying the primitiveness conditions, the maximum admissible scaling factor k is determined by:

$$k_{\max} = \left\lfloor \sqrt{\frac{R_{\max}}{2AB}} \right\rfloor.$$

Structure of the Solution Set For each r satisfying $r \equiv \rho \pmod{240}$ for some $\rho \in \mathcal{R}$, the solution set

$$\mathcal{S}(r) = \{(x^2, y^2, z^2) \in \mathbb{N}^3 : y^2 - x^2 = z^2 - y^2 = r\}$$

may have different cardinalities $|\mathcal{S}(r)|$. Classifying values of r according to the number of representations $|\mathcal{S}(r)|$ is related to the number of factorizations of $r/2$ into a product ab , where (a, b, y) forms a Pythagorean triple.

For a given r to serve as a common difference, there must exist at least three distinct arithmetic progressions of square numbers with the same common difference r , that is, $|\mathcal{S}(r)| \geq 3$.

If $|\mathcal{S}(r)| = l \geq 3$, then the number of possible configurations of three progressions is:

$$\binom{l}{3} = \frac{l!}{3!(l-3)!}.$$

For each such configuration, we calculate the differences $d_1 = A_2 - A_1$ and $d_2 = A_3 - A_2$, adhering to the ordering convention $A_1 < A_2 < A_3$ described in Section 2.3. We then calculate the percentage error p defined in that section. The optimization objective is to find a combination for which p is minimized. In the ideal case, $p = 0\%$, corresponding to a magic square of square numbers.

In summary, the analytical and computational framework formulated in this section provides the tools for deterministically eliminating successive classes of integers, through systematic iteration over values of r , that cannot generate a magic square for structural reasons. At the same time, introducing the percentage error p provides the model with a continuous evaluation scale, making it possible to analyze precisely how close the examined configurations come to an ideal solution ($p = 0\%$) within a prescribed range. However, the most important feature of this methodology is the complete disjointness of the defined subproblems. This property, which cannot be achieved using a conventional brute-force algorithm, eliminates overlap between the search spaces and enables conflict-free computation, including in distributed environments.

2.4.2 Results of the Finite-Range Exploration of Differences r and Statistical Characteristics of the Error p

The deterministic strategy described above was subjected to comprehensive empirical verification by numerically searching a finite interval of values of the common-difference parameter $r \in [2, 70\,000\,000\,000\,000]$. To ensure maximum computational efficiency and full numerical precision, the algorithm was implemented in Python using multiprocessing.

Using the mathematical framework developed earlier, a structural classification of the solution space was performed according to its cardinality. The algorithm identified the exact number of distinct values of r that generate a solution set of a given cardinality $|\mathcal{S}(r)|$, reflecting the number of available three-term arithmetic progressions of square numbers for a given r :

- Number of values of r for which $|\mathcal{S}(r)| = 1$: 27,216,984,
- Number of values of r for which $|\mathcal{S}(r)| = 2$: 929,290,
- Number of values of r for which $|\mathcal{S}(r)| = 3$: 280,109,
- Number of values of r for which $|\mathcal{S}(r)| = 4$: 12,564,
- Number of values of r for which $|\mathcal{S}(r)| = 5$: 1,549,
- Number of values of r for which $|\mathcal{S}(r)| = 6$: 55,
- Number of values of r for which $|\mathcal{S}(r)| = 7$: 4,
- Number of values of r for which $|\mathcal{S}(r)| \geq 8$: 0.

Within the investigated range, the smallest value of r for which at least one progression exists ($|\mathcal{S}(r)| \geq 1$) was found to be $r = 24$, whereas the largest value recorded among all generated solutions was $r = 69\,999\,993\,819\,360$.

According to the adopted methodology, a necessary condition for performing a combinatorial analysis and calculating the percentage error p is that the solution set has cardinality $|\mathcal{S}(r)| \geq 3$. The total number of distinct values of r satisfying this criterion and generating at least three independent three-term progressions was exactly 294,281 (the sum of the cases with cardinalities 3, 4, 5, 6, and 7).

The optimization process aimed at determining the percentage error function was based on the 294 281 distinct values of r for which $|\mathcal{S}(r)| \geq 3$. Within this set, a subspace consisting of 347 095 admissible combinations of triples of arithmetic progressions of square numbers was identified. These combinations were evaluated in terms of the location, dispersion, and skewness of the error p :

Minimum error obtained:	0.250989%,
Mean error:	113.807895%,
Median:	110.175438%,
Maximum error obtained:	199.998731%,
Standard deviation:	48.177108,
Distribution skewness:	− 0.427631,
Kurtosis:	− 0.156898.

Furthermore, the distribution of the error p was analyzed using the calculated quartiles and percentile thresholds:

- Lower quartile (25% of results have an error below this value): 110.175439%,
- Upper quartile (75% of results have an error below this value): 152.751066%,
- Interquartile range (IQR): 42.575627%,
- Threshold for the TOP 10% of best results: 36.972345%,
- Threshold for the TOP 5% of best results: 12.473291%,
- Threshold for the TOP 1% of best results: 12.473291%.

The analysis showed that the global minimum of the percentage error function within the investigated range ($p = 0.250989\%$) was attained at the original value of the common-difference parameter $r = 85\,878\,334\,080$, with internal differences $d_1 = 175\,412\,460\,288$ and $d_2 = 175\,853\,280\,285$. The optimal configuration for this value of r consists of a combination of three arithmetic progressions of square numbers, ordered increasingly according to the values of their first terms, $A_1 < A_2 < A_3$:

- Progression 1:** (19 119 699 076, 104 998 033 156, 190 876 367 236),
Progression 2: (194 532 159 364, 280 410 493 444, 366 288 827 524),
Progression 3: (370 385 439 649, 456 263 773 729, 542 142 107 809).

The statistical analysis of the percentage error p revealed substantial variation in the quality of fit among the analyzed combinations of triples of arithmetic progressions of

square numbers. The high mean (113.81%) and median (110.18%), together with a standard deviation of 48.18, indicate considerable variability in the results and a predominance of solutions characterized by relatively high errors. At the same time, the minimum error of 0.251% confirms the existence of combinations that provide a very high degree of agreement. The negative skewness coefficient (-0.428) indicates moderate left-skewness in the distribution, suggesting the presence of results with relatively low errors.

3 Eisenstein Triangles and Dimensionality Reduction of the Near-Miss Problem

To demonstrate the existence of specific classes of solutions for which the percentage error function p asymptotically approaches 0%, it is necessary to move beyond the numerical approach described above and undertake a rigorous structural analysis. To this end, we exploit the fact that the search for three-term arithmetic progressions of square numbers is also closely related to a classical problem in number theory known as the *congruum problem*.

Every three-term arithmetic progression of square numbers $\alpha^2, \beta^2, \gamma^2$ can be parameterized using two integers u, v ($u \neq v$, $u, v \neq 0$) as follows:

$$\begin{aligned}\alpha^2 &= (u^2 - 2uv - v^2)^2, \\ \beta^2 &= (u^2 + v^2)^2, \\ \gamma^2 &= (u^2 + 2uv - v^2)^2.\end{aligned}$$

The common difference of this progression is given by

$$r = 4uv(u^2 - v^2)[16].$$

The set of eight parameter pairs

$$\Omega = \{(\pm u, \pm v), (\pm v, \pm u)\}$$

forms an orbit under the action of the dihedral group D_4 on the parameter space of the generators. The transformations corresponding to these parameter pairs preserve the set of values of the progression terms, potentially changing only their order or orientation.

Proof. The group D_4 is generated by two transformations of the parameter space: reflection across the line $u = v$, corresponding to the exchange of variables $\sigma(u, v) = (v, u)$, and a sign change of one variable, given by $\tau(u, v) = (-u, v)$. Consider the effect of these generators on the progression's difference function $r(u, v) = 4uv(u^2 - v^2)$:

$$\begin{aligned}r(\sigma(u, v)) &= r(v, u) = 4vu(v^2 - u^2) = -4uv(u^2 - v^2) = -r(u, v), \\ r(\tau(u, v)) &= r(-u, v) = 4(-u)v((-u)^2 - v^2) = -4uv(u^2 - v^2) = -r(u, v).\end{aligned}$$

Since the terms $\alpha^2, \beta^2, \gamma^2$ in the progression parameterization depend on u^2 , v^2 , and the square of the product uv in the final expressions, changing the sign of the common difference, $r \mapsto -r$, corresponds only to reversing the order of the progression terms (changing its orientation) or to passing to an isomorphic configuration. The orbit under the group generated by these transformations contains exactly eight elements and takes the form of the set Ω . $\square$

The three desired arithmetic progressions can therefore be expressed parametrically using the pairs (a, b) , (c, d) , and (e, f) . This yields:

• **Progression 1:**

$$\begin{aligned} A &= (a^2 - 2ab - b^2)^2, \\ A + r &= (a^2 + b^2)^2, & r_1 &= 4ab(a^2 - b^2). \\ A + 2r &= (a^2 + 2ab - b^2)^2, \end{aligned}$$

• **Progression 2:**

$$\begin{aligned} A + d_0 &= (c^2 - 2cd - d^2)^2, \\ A + d_0 + r &= (c^2 + d^2)^2, & r_2 &= 4cd(c^2 - d^2). \\ A + d_0 + 2r &= (c^2 + 2cd - d^2)^2, \end{aligned}$$

• **Progression 3:**

$$\begin{aligned} A + 2d_0 &= (e^2 - 2ef - f^2)^2, \\ A + 2d_0 + r &= (e^2 + f^2)^2, & r_3 &= 4ef(e^2 - f^2). \\ A + 2d_0 + 2r &= (e^2 + 2ef - f^2)^2, \end{aligned}$$

According to the theoretical foundations of the problem reduction presented in Section 2.1, the search for a magic square of square numbers reduces to finding three arithmetic progressions with a common difference, $r_1 = r_2 = r_3 = r$, whose first terms themselves form an arithmetic progression with common difference d_0 . The parameterization introduced above therefore imposes two algebraic constraints on the generator pairs (a, b) , (c, d) , and (e, f) .

The first constraint follows from the requirement that all three progressions have the same internal common difference. It takes the form

$$ab(a^2 - b^2) = cd(c^2 - d^2) = ef(e^2 - f^2). \quad (\text{E8})$$

The second constraint follows from the requirement that the first terms of the three progressions form an arithmetic progression. The equality of the successive differences between these terms can be expressed as

$$(c^2 - 2cd - d^2)^2 - (a^2 - 2ab - b^2)^2 = (e^2 - 2ef - f^2)^2 - (c^2 - 2cd - d^2)^2. \quad (\text{E9})$$

In addition to the algebraic relations above, an independent structural constraint must be imposed: the parameter pairs (a, b) , (c, d) , and (e, f) must belong to distinct orbits under the action of the dihedral group D_4 on the generator space.

The equations above, together with the orbit-disjointness condition, constitute a complete system of constraints that the parameters (a, b) , (c, d) , and (e, f) must satisfy for the corresponding three arithmetic progressions of square numbers to form a 3×3 magic square composed of square numbers.

Directly searching for solutions to this system in the original six-dimensional parameter space is analytically and numerically inefficient. To overcome this structural obstacle, we propose replacing conventional discrete exploration with a rigorous geometric parameterization based on the properties of Eisenstein triangles with an angle of 120° . As we will demonstrate below, this structure makes it possible to satisfy condition (E8) automatically and simultaneously, reducing the entire problem to the study of the behavior of a single variable.

3.1 Geometric Parameterization of Eisenstein Triangles

An Eisenstein triangle with an angle of 120° is defined as a triangle with integer side lengths X, Y, U , in which the angle between the sides X and Y is 120° . By the law of cosines and the fact that $\cos(120^\circ) = -\frac{1}{2}$, this relationship takes the form of a Diophantine equation:

$$\begin{aligned} U^2 &= X^2 + Y^2 - 2XY \cos(120^\circ) \\ U^2 &= X^2 + XY + Y^2. \end{aligned} \tag{E10}$$

The positive integer solutions to this equation can be parameterized using two positive integers m, n ($m > n > 0$), as follows:

$$\begin{aligned} X &= m^2 - n^2, \\ Y &= 2mn + n^2, \\ U &= m^2 + mn + n^2 [17, 18]. \end{aligned}$$

3.2 Construction of Three Pairs of Generators

The structure of an Eisenstein triangle makes it possible to construct three pairs of generators corresponding to the three desired arithmetic progressions. We adopt the following substitutions:

$$(a, b) = (U, X), \quad (c, d) = (U, Y), \quad (e, f) = (X + Y, U).$$

To demonstrate that these pairs yield identical internal common differences, we introduce the auxiliary function $f(x, y) = xy(x^2 - y^2)$. Direct substitution of the defined parameters and application of identity (E10) yield the following relations:

$$\begin{aligned} f(U, X) &= UX(U^2 - X^2) = UX(XY + Y^2) = UXY(X + Y), \\ f(U, Y) &= UY(U^2 - Y^2) = UY(X^2 + XY) = UXY(X + Y). \end{aligned}$$

Similarly, for the third pair of generators, expanding the square $(X + Y)^2 = X^2 + 2XY + Y^2$ gives:

$$\begin{aligned} f(X + Y, U) &= (X + Y)U [(X + Y)^2 - U^2] \\ &= (X + Y)U [X^2 + 2XY + Y^2 - (X^2 + XY + Y^2)] \\ &= UXY(X + Y). \end{aligned}$$

The value of the function f is therefore identically equal for all three parameter pairs, proving that the internal common differences coincide and yielding the explicit expression for the common difference r :

$$f(U, X) = f(U, Y) = f(X + Y, U) = UXY(X + Y) \implies r = 4UXY(X + Y).$$

It should be emphasized that the adopted geometric parameterization automatically guarantees that condition (E8)—the equality of the internal common differences of the three progressions—is satisfied. Condition (E9), which requires the first terms to form an arithmetic progression, is not satisfied automatically. Relaxing this latter condition is precisely the central idea behind the near-miss approach described in the preceding sections.

3.3 Explicit Forms of the Progressions and the Error Function

Using the parameterization of triples of squares and the auxiliary variables (X, Y, U) introduced above, the explicit terms of the three desired arithmetic progressions take the following form:

$$\textbf{Progression 1: } (U^2 - 2UX - X^2)^2, \quad (U^2 + X^2)^2, \quad (U^2 + 2UX - X^2)^2,$$

$$\textbf{Progression 2: } (U^2 - 2UY - Y^2)^2, \quad (U^2 + Y^2)^2, \quad (U^2 + 2UY - Y^2)^2,$$

$$\textbf{Progression 3: } [(X + Y)^2 - 2(X + Y)U - U^2]^2, \\ [(X + Y)^2 + U^2]^2, \quad [(X + Y)^2 + 2(X + Y)U - U^2]^2.$$

Under the adopted geometric parameterization, the terms of the three arithmetic progressions under consideration take explicit polynomial forms in the variables m and n . Following the ordering convention $A_1 < A_2 < A_3$ established in Section 2.3, the differences $d_1 = A_2 - A_1$ and $d_2 = A_3 - A_2$ are defined as the differences between the first terms of successive progressions. Substituting these expressions, $d_1(m, n)$ and $d_2(m, n)$, into the general formula for the percentage error p defined in Section 2.3, and then expanding the powers and collecting like terms, yields a ratio of two homogeneous polynomials of degree eight:

$$p(m, n) = \frac{200 \cdot |\mathcal{L}(m, n)|}{\mathcal{M}(m, n)},$$

where the numerator $\mathcal{L}(m, n)$ and denominator $\mathcal{M}(m, n)$ are given by

$$\begin{aligned} \mathcal{L}(m, n) &= -6m^8 - 24m^7n - 36m^6n^2 - 24m^5n^3 + 60m^4n^4 \\ &\quad + 132m^3n^5 + 102m^2n^6 + 36mn^7 + 3n^8, \\ \mathcal{M}(m, n) &= 16m^7n + 56m^6n^2 + 80m^5n^3 + 60m^4n^4 \\ &\quad + 28m^3n^5 + 10m^2n^6 - 4mn^7 - 3n^8. \end{aligned}$$

3.4 Reduction to a Function of One Variable

Expressing $p(m, n)$ as the ratio of two homogeneous polynomials of degree eight has an important structural consequence. Since both the numerator $\mathcal{L}$ and the denominator $\mathcal{M}$ are homogeneous, for any constant $\lambda \neq 0$ we have

$$\mathcal{L}(\lambda m, \lambda n) = \lambda^8 \mathcal{L}(m, n), \quad \mathcal{M}(\lambda m, \lambda n) = \lambda^8 \mathcal{M}(m, n).$$

Consequently, the factor λ^8 cancels completely in the ratio, which means that the percentage error satisfies the identity $p(\lambda m, \lambda n) = p(m, n)$. The percentage error therefore depends only on the ratio of the variables, $k = \frac{m}{n}$, rather than on their absolute magnitudes.

This makes it possible to formally reduce the two-dimensional search space to a one-dimensional space by introducing the new dimensionless variable k . In this formulation, the values of the function $p(k)$ lie in the interval $[0\%, 200\%)$, where $p = 0\%$ corresponds to an exact magic square of square numbers—that is, the case in which $d_1 = d_2$ and the first terms of the three progressions form a perfect arithmetic progression. Ultimately, the error optimization problem reduces to studying the behavior of the single-variable function $p(k)$. Finding an ideal configuration with zero error, $p(k) = 0$, is equivalent to determining the roots of the degree-eight polynomial that forms the numerator of this function.

The set of eight algebraic solutions to this equation is given by

$$\begin{aligned}
k_1 &= -\frac{1}{2}\sqrt{-4\sqrt{3} - 2\sqrt{-2\sqrt{3} + 10} + 1} - \frac{1}{2} \approx -0.5000 - 1.6614i, \\
k_2 &= \frac{1}{2}\sqrt{-4\sqrt{3} - 2\sqrt{-2\sqrt{3} + 10} + 1} - \frac{1}{2} \approx -0.5000 + 1.6614i, \\
k_3 &= -\frac{1}{2}\sqrt{-4\sqrt{3} + 2\sqrt{-2\sqrt{3} + 10} + 1} - \frac{1}{2} \approx -0.5000 - 0.4514i, \\
k_4 &= \frac{1}{2}\sqrt{-4\sqrt{3} + 2\sqrt{-2\sqrt{3} + 10} + 1} - \frac{1}{2} \approx -0.5000 + 0.4514i, \\
k_5 &= -\frac{1}{2}\sqrt{4\sqrt{3} - 2\sqrt{2\sqrt{3} + 10} + 1} - \frac{1}{2} \approx -0.8839, \\
k_6 &= \frac{1}{2}\sqrt{4\sqrt{3} - 2\sqrt{2\sqrt{3} + 10} + 1} - \frac{1}{2} \approx -0.1161, \\
k_7 &= -\frac{1}{2}\sqrt{4\sqrt{3} + 2\sqrt{2\sqrt{3} + 10} + 1} - \frac{1}{2} \approx -2.4536, \\
k_8 &= \frac{1}{2}\sqrt{4\sqrt{3} + 2\sqrt{2\sqrt{3} + 10} + 1} - \frac{1}{2} \approx 1.4536.
\end{aligned}$$

Constructing magic squares of square numbers using the geometry of Eisenstein triangles with an angle of 120° requires the side lengths X, Y, U to be positive integers. The parameterization condition $m > n > 0$ implies that admissible values of the dimensionless variable must satisfy $k > 1$. An analysis of the resulting roots allows us to exclude the complex solutions (k_1, k_2, k_3, k_4) and the negative real solutions (k_5, k_6, k_7) . Consequently, the only mathematically and geometrically admissible candidate for an optimal point at which the error function vanishes is the positive real root

$$k_8 = \frac{1}{2}\sqrt{4\sqrt{3} + 2\sqrt{2\sqrt{3} + 10} + 1} - \frac{1}{2}.$$

Achieving a state in which the error function takes the value $p(k) = 0\%$ requires the parameter k to equal the root k_8 exactly. Since the generator parameters m and n are integers by definition, this condition can be satisfied in the discrete domain only if the optimal point is rational ($k_8 \in \mathbb{Q}$). To investigate this possibility, we formulate the following theorem concerning the algebraic properties of the optimal point.

Theorem 1. *The algebraic number defined by*

$$k_8 = \frac{1}{2}\sqrt{4\sqrt{3} + 2\sqrt{2\sqrt{3} + 10} + 1} - \frac{1}{2}$$

is irrational ($k_8 \notin \mathbb{Q}$).

Proof. The proof of irrationality is based on the rational root theorem for polynomials with integer coefficients. The number k_8 is a root of the degree-eight polynomial $W(k) \in \mathbb{Z}[k]$ obtained from the numerator of the error function:

$$W(k) = 6k^8 + 24k^7 + 36k^6 + 24k^5 - 60k^4 - 132k^3 - 102k^2 - 36k - 3 = 0.$$

If a polynomial with integer coefficients has a rational root in reduced form $\frac{s}{t}$, where $\gcd(s, t) = 1$, then the numerator s must divide the constant term $a_0 = -3$, and the denominator t must divide the leading coefficient $a_n = 6$. The possible divisors are therefore

$$\begin{aligned}s &\in \{\pm 1, \pm 3\}, \\ t &\in \{1, 2, 3, 6\},\end{aligned}$$

which yields the following finite set of candidates:

$$\mathbb{Q}_{\text{candidates}} = \left\{ \pm 1, \pm 3, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{1}{6} \right\}.$$

Direct substitution shows that $W(\frac{s}{t}) \neq 0$ for every element of $\mathbb{Q}_{\text{candidates}}$, meaning that none of the candidates is a root of the polynomial $W(k)$. Since any rational root would have to belong to this set, it follows that the optimal point is irrational ($k_8 \in \mathbb{R} \setminus \mathbb{Q}$). $\square$

The irrationality of the optimal point k_8 implies that the state $p = 0\%$ cannot be attained for any finite integer values of the parameters m and n . However, since the continued fraction expansion of an irrational algebraic number of higher degree is infinite, it provides a sequence of increasingly accurate rational approximations.

Let $[a_0; a_1, a_2, a_3, \dots]$ be the infinite continued fraction expansion of k_8 . The sequence of its convergents is defined by

$$\frac{P_i}{Q_i} = [a_0; a_1, a_2, \dots, a_i], \quad \text{for } i \in \mathbb{N},$$

where P_i and Q_i are relatively prime positive integers. Taking each convergent as the ratio of the generator parameters gives

$$m_i = P_i, \quad n_i = Q_i.$$

The convergents of an infinite continued fraction satisfy the standard approximation bound

$$\left| k_8 - \frac{P_i}{Q_i} \right| < \frac{1}{Q_i^2}.$$

Using the notation introduced above, namely $m_i = P_i$ and $n_i = Q_i$, we obtain

$$\left| k_8 - \frac{m_i}{n_i} \right| < \frac{1}{n_i^2}.$$

Since the denominators of successive convergents satisfy $n_i \rightarrow \infty$, the right-hand side of this inequality tends to zero. Hence,

$$\left| k_8 - \frac{m_i}{n_i} \right| \longrightarrow 0,$$

which is equivalent to

$$\lim_{i \rightarrow \infty} \frac{m_i}{n_i} = k_8.$$

Consequently, the ratios $\frac{m_i}{n_i}$ form a sequence of increasingly accurate approximations to k_8 .

The percentage error function, expressed in terms of the variable $k = \frac{m}{n}$, is continuous in a neighborhood of k_8 . By the continuity of the function and the fact that $p(k_8) = 0\%$, we have

$$\lim_{i \rightarrow \infty} p(m_i, n_i) = \lim_{i \rightarrow \infty} p\left(\frac{m_i}{n_i}\right) = p\left(\lim_{i \rightarrow \infty} \frac{m_i}{n_i}\right) = p(k_8) = 0\%.$$

This means that continued fractions make it possible to construct an infinite sequence of *near-miss* configurations for which the magic-sum error approaches zero asymptotically as the parameter n increases. The numerical behavior of the error function p for the first ten convergents approximating the optimal point $k_8 \approx 1.4536$ is presented in Table 1.

Table 1: Values of the parameters m and n and the corresponding percentage error p for successive convergents of the continued fraction expansion of k_8 .

Index (i)	m	n	p [%]
0	1	1	(Omitted: $m \leq n$)
1	3	2	12.4733
2	13	9	2.58057
3	16	11	0.250989
4	141	97	0.00989004
5	298	205	0.00411786
6	439	302	0.000381250
7	3371	2319	0.0000164761
8	10552	7259	$7.06660 \cdot 10^{-8}$
9	784219	539485	$4.61145 \cdot 10^{-10}$
10	1578990	1086229	$1.41809 \cdot 10^{-11}$

3.5 Polynomial Matrix Structure and Asymptotic Convergence of the Diagonals

The irrationality of the optimal point k_8 , established in the previous section, has direct implications for the algebraic structure of the matrix generated by the geometry of Eisenstein triangles. Substituting the explicit polynomial expressions for the component arithmetic progressions into the square grid yields a general matrix parameterized by the original variables m and n . To simplify the notation, let us define the entries of this matrix as homogeneous polynomials of degree eight, denoted by $P_{ij}(m, n)$:

$$\mathcal{M}(m, n) = \begin{pmatrix} P_{11}(m, n) & P_{12}(m, n) & P_{13}(m, n) \\ P_{21}(m, n) & P_{22}(m, n) & P_{23}(m, n) \\ P_{31}(m, n) & P_{32}(m, n) & P_{33}(m, n) \end{pmatrix},$$

where the individual entries have the following explicit algebraic forms:

$$\begin{aligned}
P_{11} &= 4m^8 + 8m^7n + 8m^6n^2 + 12m^5n^3 + 17m^4n^4 + 12m^3n^5 + 8m^2n^6 + 8mn^7 + 4n^8, \\
P_{12} &= 4m^8 + 32m^7n + 92m^6n^2 + 120m^5n^3 + 77m^4n^4 + 12m^3n^5 - 10m^2n^6 - 4mn^7 + n^8, \\
P_{13} &= m^8 - 4m^7n - 10m^6n^2 + 12m^5n^3 + 77m^4n^4 + 120m^3n^5 + 92m^2n^6 + 32mn^7 + 4n^8, \\
P_{21} &= 4m^8 + 16m^7n + 36m^6n^2 + 64m^5n^3 + 77m^4n^4 + 68m^3n^5 + 46m^2n^6 + 12mn^7 + n^8, \\
P_{22} &= m^8 + 4m^7n + 18m^6n^2 + 40m^5n^3 + 77m^4n^4 + 92m^3n^5 + 64m^2n^6 + 24mn^7 + 4n^8, \\
P_{23} &= 4m^8 + 16m^7n + 36m^6n^2 + 40m^5n^3 + 17m^4n^4 - 16m^3n^5 - 20m^2n^6 + 4n^8, \\
P_{31} &= m^8 + 12m^7n + 46m^6n^2 + 68m^5n^3 + 77m^4n^4 + 64m^3n^5 + 36m^2n^6 + 16mn^7 + 4n^8, \\
P_{32} &= 4m^8 - 20m^6n^2 - 16m^5n^3 + 17m^4n^4 + 40m^3n^5 + 36m^2n^6 + 16mn^7 + 4n^8, \\
P_{33} &= 4m^8 + 24m^7n + 64m^6n^2 + 92m^5n^3 + 77m^4n^4 + 40m^3n^5 + 18m^2n^6 + 4mn^7 + n^8.
\end{aligned}$$

A structural analysis of the matrix $\mathcal{M}(m, n)$ leads to the following conclusion: for all integer parameters satisfying the model's domain condition $m > n > 0$, the array consistently remains a semimagic square. This means that exactly seven of its eight characteristic sums are identically equal. The magic-sum condition is satisfied exactly for all three rows, all three columns, and the descending main diagonal (D_1):

$$\begin{aligned}
\sum_{j=1}^3 P_{ij} &= \sum_{i=1}^3 P_{ij} = D_1(m, n) = 9m^8 + 36m^7n + 90m^6n^2 + 144m^5n^3 + 171m^4n^4 \\
&\quad + 144m^3n^5 + 90m^2n^6 + 36mn^7 + 9n^8.
\end{aligned}$$

The only element that prevents the structure from being a perfect magic square is the sum along the ascending diagonal (D_2), defined as $D_2(m, n) = P_{13} + P_{22} + P_{31}$. The sum of the polynomials corresponding to these three entries is

$$\begin{aligned}
D_2(m, n) &= 3m^8 + 12m^7n + 54m^6n^2 + 120m^5n^3 + 231m^4n^4 \\
&\quad + 276m^3n^5 + 192m^2n^6 + 72mn^7 + 12n^8.
\end{aligned}$$

Consequently, the deviation from perfect combinatorial structure depends solely on the discrepancy between the descending diagonal sum $D_1(m, n)$ and the ascending diagonal sum $D_2(m, n)$. A perfect magic square of square numbers within this geometric model would exist if and only if the Diophantine equation $D_1(m, n) = D_2(m, n)$ were satisfied. After collecting like terms, this equation reduces directly to finding the roots of the numerator polynomial of the error function, $\mathcal{L}(m, n) = 0$, defined in Section 3.3.

This would require the generator parameters to satisfy the exact proportionality relation $m = k_8 \cdot n$. Since $k_8 \notin \mathbb{Q}$, equality of the two diagonal sums is structurally impossible over the integers. Nevertheless, Diophantine approximation using successive continued-fraction convergents, listed in Table 1, makes it possible to generate pairs (m_i, n_i) that provide increasingly accurate rational approximations to this ratio. As the order of approximation increases, the values of the two diagonal sums, D_1 and D_2 , become increasingly close, and the percentage error of the near-miss structure converges to zero.

To illustrate the convergence of the approximation method, let us extend the numerical analysis to the continued-fraction convergent with index $i = 24$. For $i = 24$, the generator parameters are

$$m_{24} = 573\,745\,860\,893, \quad n_{24} = 394\,694\,958\,633.$$

These parameters yield a semimagic square structure represented by the matrix $\mathcal{M}_{573745860893,394694958633}$ of square numbers:

$$\begin{pmatrix} 536190346598695687381956191974460797017088253249^2 & 1218346200561723778955236543587748766902562991938^2 & 730461692335687908331884958424865542532884746658^2 \\ 1007251114789090707693697941142585504420498294898^2 & 876630259607537286005516213248543057744382745178^2 & 722776878348560933616142305214383853855940388689^2 \\ 1001692038577346015011841473855226741714123718898^2 & 229333293910780370310934252764141866779887567409^2 & 1117792975614682874976263772186305288576034579818^2 \end{pmatrix}$$

For this matrix, the diagonal sums are

$$D_1 = 2305441836178734654406338345720287610155827736288605412333198986020133449296017489229937054220809$$

$$D_2 = 2305441836178734654406341182353462411854855052565918500978487084762342152533869788832205846755052$$

A comparison of the two numerical values shows that their first 23 significant digits coincide, while the error is $p_{(m,n)} = 5.8976 \cdot 10^{-22}$.

It is worth noting that Frank Rubin investigated semimagic 3×3 squares of square numbers in which all row sums, column sums, and one diagonal sum are equal, while the ratio of the two diagonal sums, $(D_1 : D_2)$, is as close to 1 as possible. In 2009, he proposed a semimagic square of square numbers of order three in which the ratio of the two diagonal sums approximated 1 from above, with a value of approximately $1 + 7.057898 \times 10^{-17}$ [13].

The method proposed above allows us to find, relatively quickly, semimagic squares for which the ratio of the two diagonal sums provides an even closer approximation to 1. For example, at the continued-fraction convergent with index $i = 55$, the ratio of the diagonal sums is approximately $1 + 1.099096394 \times 10^{-53}$, with an error of $p_{(m,n)} = 5.2682 \cdot 10^{-51}$. For $i = 2500$, the ratio of the diagonal sums approximates 1 from below and reaches $1 - 2.651711035 \times 10^{-2570}$, with an error of $p_{(m,n)} = 1.27102 \cdot 10^{-2567}$. The numbers of digits in the generators m_i and n_i , together with the corresponding errors and ratios of the two diagonals for selected indices i , are presented in Table 2.

Table 2: Comparison of the number of digits in the generators, error values, and diagonal ratios for higher-order approximations.

Index (i)	Digits in m	Digits in n	Error p	Ratio D_1/D_2
24	12	12	$5,8976 \cdot 10^{-22}$	$\approx 1 - 1,2304 \cdot 10^{-24}$
55	27	27	$5,2682 \cdot 10^{-51}$	$\approx 1 + 1,0991 \cdot 10^{-53}$
100	51	50	$1,9795 \cdot 10^{-99}$	$\approx 1 - 4,1298 \cdot 10^{-102}$
2500	1285	1285	$1,2710 \cdot 10^{-2567}$	$\approx 1 - 2,6517 \cdot 10^{-2570}$
5000	2537	2537	$9,1196 \cdot 10^{-5071}$	$\approx 1 - 1,9026 \cdot 10^{-5073}$
10000	5154	5154	$3,7886 \cdot 10^{-10307}$	$\approx 1 - 7,9040 \cdot 10^{-10310}$
20000	10365	10365	$8,7689 \cdot 10^{-20728}$	$\approx 1 - 1,8294 \cdot 10^{-20730}$
50000	25957	25957	$5,6171 \cdot 10^{-51912}$	$\approx 1 - 1,1719 \cdot 10^{-51914}$
100000	51615	51615	$3,5233 \cdot 10^{-103227}$	$\approx 1 - 7,3506 \cdot 10^{-103230}$

4 Summary and Conclusions

This paper presents a structural exploration of the problem of a 3×3 magic square of square numbers. Although the original problem formulated by Martin LaBar remains open, the methodology adopted in this work has enabled a systematic investigation of *near-miss* structures, both numerically and analytically.

The principal results and conclusions obtained in the course of this research are as follows:

1. **Effectiveness of modular filtering:** The use of modular arithmetic in the ring $\mathbb{Z}_{240}$ proved to be an effective tool for reducing the search space. Defining a set of admissible residue classes made it possible to immediately eliminate 97.5% of structurally inconsistent residue classes, thereby reducing the computational cost of the deterministic algorithm.
2. **Numerical exploration of the common-difference range:** The custom algorithm implemented in Python enabled an exhaustive search over a broad, finite range of common differences, $r \in [2, 70\,000\,000\,000\,000]$. Within the investigated range, the global minimum of the percentage error function was found to be $p \approx 0.250989\%$, attained at $r = 85\,878\,334\,080$. This identified the numerically optimal configuration of the three arithmetic progressions.
3. **Dimensionality reduction through Eisenstein geometry:** Employing the geometric parameterization of Eisenstein triangles with an angle of 120° made it possible to satisfy automatically the condition that the internal differences of the progressions under consideration are equal. This transformation reduced the two-dimensional optimization problem to the study of a function of a single dimensionless variable, $k = \frac{m}{n}$, yielding an explicit polynomial matrix whose seven line sums are identical for arbitrary integer parameter values, thereby preserving its semimagic-square property.
4. **Proof of the irrationality of the optimal point:** It was shown that the only geometrically admissible point corresponding to the theoretical optimum, k_8 , at which the error function vanishes, is an algebraic number of degree eight. The proof of its irrationality constitutes a key structural conclusion: within the geometric model considered, it is impossible to construct an exact 3×3 magic square of square numbers using integer parameters.
5. **Asymptotic convergence toward ideal structures:** The application of Diophantine approximation and the continued-fraction expansion of k_8 provided a systematic method for generating an infinite sequence of rational approximations. The resulting parameters make it possible to construct semimagic squares with arbitrarily small percentage errors. This is illustrated by the numerical example corresponding to the convergent with index $i = 100\,000$, for which the ratio of the diagonal sums is $1 - 7,3506 \cdot 10^{-103230}$.

The research framework developed in this paper, which combines computational techniques with number theory and Diophantine geometry, provides a general tool for analyzing *near-miss* structures. Although this work does not rule out the existence of a perfect 3×3 magic square of square numbers arising from other parameter configurations, the continued-fraction strategy presented here establishes a mathematically rigorous approach to generating semimagic structures with a degree of precision not previously reported in the literature.

References

- [1] Schwartzman, S. (1994). *The words of mathematics: An etymological dictionary of mathematical terms used in English*. MAA. p. 130.

- [2] Swetz, Frank J. (2008). *The Legacy of the Luoshu* (2nd ed.). A.K. Peters/CRC Press.
- [3] Everitt, D., Racinski, F. (2020). Creative Visualisation of Magic Squares. W: *Proceedings of EVA London 2020* (pp. 217-224). BCS Learning & Development.
- [4] Tolsa, C. (2020). The earliest Arabic magic squares. *Suhayl. Journal for the History of the Exact and Natural Sciences in Islamic Civilisation*, 7-24.
- [5] Láng, Benedek (2006). Demons in Krakow, and Image Magic in a Magical Handbook. W: G. Klaniczay, E. Pócs (red.), *Christian Demonology and Popular Mythology*. Central European University Press.
- [6] Boyer, C. (2005). Some Notes on the Magic Squares of Squares Problem. *Mathematical Intelligencer*, 27(2).
- [7] Pfeffermann, G. (1891). Les tablettes du chercheur. *Journal des Jeux d'Esprit et de Combinaisons*.
- [8] Boyer, Christian. "Bimagic squares". *Multimagie.com*. Retrieved 12 September 2026.
- [9] LaBar, Martin (1984). Problems. *The College Mathematics Journal*, 15(1), 68–74.
- [10] Robertson, John P. (1996). Magic Squares of Squares. *Mathematics Magazine*, 69(4), 289–293.
- [11] Gardner, Martin (1996). The magic of 3x3. *Quantum*, 6(3), 24–26.
- [12] Gardner, Martin (1996). The latest magic. *Quantum*. 6(4).
- [13] Boyer, Christian. "Magic squares of squares". *Multimagie.com*. Retrieved 12 September 2026.
- [14] Bremner, Andrew (1999). On squares of squares. *Acta Arithmetica*.
- [15] Trautman, Andrzej (1998). Pythagorean spinors and Penrose twistors. In: S.A. Hugget; L.J. Mason; K.P. Tod; S.T. Tsou; N.M.J. Woodhouse (eds.), *Geometric universe*.
- [16] Darling, David (2004), *The Universal Book of Mathematics: From Abracadabra to Zeno's Paradoxes*, John Wiley & Sons.
- [17] Read, E. (2006). On integer-sided triangles containing angles of 120° or 60° . *The Mathematical Gazette*, 90, 299–305.
- [18] Selkirk, K. (1983). Integer-sided triangles with an angle of 120° . *The Mathematical Gazette*, 67, 251–255.