# The General Solution to the Twin Prime Problem, or Constructivism vs. Analytical Atavism

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**Abstract**

This paper proposes a fundamentally new approach to investigating the structure of the set of natural numbers based on the concept of alternative and connected numerical sets, one of which is the set of prime numbers. The method of dynamic sieves is described. By utilizing the proposed kinematic model and the theory of connected sets, a theorem is proven on the infinity of sets of $n$-tuple primes of arbitrary length and configuration. While the theory of connected sets expands the theorem, the theory of alternative sets deepens it, rendering the theorem global.

**Keywords:** $n$-tuple primes, method of dynamic sieves, theory of connected sets, alternative sets, binding codes, constructivism in number theory.

# 1 Introduction

For a long time in number theory, the dominant perception was that an exact description of the distribution density of higher-order tuples (such as quadruplets and broader configurations) lay at the very limit of what elementary methods could achieve. However, detailed numerical analysis allows for the detection of large-scale regularities in the distribution of prime numbers within remote segments of the natural number sequence.

The situation changes fundamentally with the introduction of the concept of alternative numerical sets. Within the framework of this theory, the set of prime numbers ceases to be an isolated anomaly; rather, it proves to be merely a special case ("a drop in the ocean") of a broad class of alternative sets governed by unified, fundamental laws of density.

The pivotal step that enabled overcoming the divergence problem of infinite products of the form $\prod(p_i - n)/p_i$ was the linking together of $n$-tuple prime sets of various configurations with arbitrary internal intervals. This permitted the abandonment of calculating infinite asymptotic "tails," as they prove to be invariant for all connected sets.

The developed approach allows for the application of elementary combinatorics methods to the dynamic shifts of residue systems, thereby reducing the proof of the infinity of $n$-tuple primes of free configuration to a concise and rigorous algorithm — much like Euclid's classical proof of the infinity of the set of prime numbers. First, however, it is necessary to introduce a series of definitions.

## 2 The Concept of the Binding Code and Alternative Sets

As each natural number enters the numerical sequence, it is uniquely projected onto an infinite system of residues modulo all primes.

**Definition 1.** *The binding code (BC) of a natural number $x$ is defined as the infinite sequence of its residues modulo all prime numbers, taken in ascending order:*

$$BC(x) = (r_1, r_2, r_3, \dots), \tag{1}$$

*where $r_i \equiv x \pmod{p_i}$, and $p_i$ is the $i$-th prime number ($p_1 = 2, p_2 = 3, p_3 = 5, \dots$).*

**Definition 2.** ***The intersection of two binding codes***
*$BC(A) = (a_i)$ and $BC(B) = (b_i)$ is non-empty if and only if there exists at least one common index $i \in \mathbb{N}$ for which the coordinate values coincide:*

$$BC(A) \cap BC(B) \neq \emptyset \iff \exists i \in \mathbb{N} : (a_i = b_i) \tag{2}$$

Let us consider the initial segment of the BC based on the first four prime numbers: $P = \{2, 3, 5, 7\}$. The product of these moduli defines the primorial of the interval:

$$P_4\# = 2 \cdot 3 \cdot 5 \cdot 7 = 210$$

According to the Chinese Remainder Theorem, within any interval of magnitude 210, there exist exactly 210 unique variations of the BC. Let us demonstrate the fundamental harmony of the natural number sequence within this interval:

1. The number of BCs in the range of 210 that do not contain a residue of 0 (i.e., are not divisible by 2, 3, 5, 7) equals: $210 \cdot 1/2 \cdot 2/3 \cdot 4/5 \cdot 6/7 = 48$. This is the prototype of the set of prime numbers.

2. The number of BCs that fundamentally do not contain a residue of 1 at the same positions is calculated analogously: $210 \cdot 1/2 \cdot 2/3 \cdot 4/5 \cdot 6/7 = 48$.

3. The general law of non-intersection: for any arbitrarily chosen fixed BC within the interval of 210, there exist exactly 48 other BCs that do not intersect it at any element (residue). These rules are invariant for any interval of magnitude 210.

**Prime numbers** differ only in that their BC does not intersect with the BC(0), which consists entirely of zeros. However, this set is opposed by an infinite class of sets whose BCs do not intersect with any (arbitrarily chosen) BC.

**Definition 3.** ***(BC of a Set)*** *The binding code of a set $A$ (denoted as $BC(A)$) is a set consisting of the binding codes of the individual elements belonging to this set:*

$$BC(A) = \{BC(x) \mid x \in A\} \tag{3}$$

**Definition 4.** *(**Intersection Condition between the BC of a Set and the BC of a Number**). The binding code of a set $A$ intersects with the binding code of a specific number $\omega$ if and only if there exists at least one element $a_i$ within the set $A$ whose individual binding code has a non-empty intersection with the code of the number $\omega$:*

$$BC(A) \cap BC(\omega) \neq \emptyset \iff \exists a_i \in A : (BC(a_i) \cap BC(\omega) \neq \emptyset) \tag{4}$$

**Definition 5.** *(**Alternative Sets**). Numerical sets are called alternative if their binding codes are defined within a common domain of BC, they possess equal asymptotic densities, but their structures are constrained by non-intersection conditions with different binding codes of numbers (alternatives):*

$$BC(A) \cap BC(\omega) = \emptyset, \quad \textit{where } \omega \textit{ is an individual alternative code.} \tag{5}$$

*Furthermore, the elements of alternative sets invariantly obey a single global density distribution law:*

$$\mu(A) = \prod_{i=1}^{\infty} \frac{p_i - 1}{p_i} \tag{6}$$

*where $\mu(A)$ is the asymptotic density of the set, and $p_i$ is the $i$-th prime number.*

Let an alternative be given:

$$BC(\Omega) = (\omega_1, \omega_2, \omega_3, \ldots, \omega_i, \ldots), \quad \omega_i \in \mathbb{Z}_{p_i} \tag{7}$$

A change in even a single sign $\omega_k \to \omega'_k$ generates a new alternative and a new alternative set $A'$, for which:

$$BC(A') \cap BC(\Omega') = \emptyset \tag{8}$$

# 3 The Concept of Connected Sets

**Definition 6.** *(Connected Sets). Two infinite subsets of natural numbers are called connected if their asymptotic densities on any invariant interval of the number line are related via a constant finite coefficient $k > 0$:*

$$\mu(A_1) = k \cdot \mu(A_2), \quad \textit{where } k \in \mathbb{R}^+,\ k \neq \infty. \tag{9}$$

*Such sets function invariantly, like the gears of a single mechanism: the dynamics of their distribution are interdependent. For example, the sets of numbers that are multiples of 5 and 101 are connected by a finite coefficient $k = 101/5$.*

Within the framework of regular prime tuples, the properties of connected sets are revealed as follows:

- **Stability under Power Shifts ($k = 1$):** Twin pairs with an interval of two units $(a, a+2)$ and pairs with an interval of eight $(a, a+8)$ possess an identical residue structure modulo small primes. They exhibit equal density ($k = 1$) and are connected. To ensure equivalent initial conditions (the "start") for configurations with different internal intervals, it is necessary to align their centers of symmetry in the initial state of the model.
- **Linear Scaling of Density ($k = 2$):** Pairs with an interval of two $(a, a+2)$ and pairs with an interval of six $(a, a+6)$ possess different densities. The modulus $p = 3$ blocks two residue classes for the interval of 2 (the survival fraction is $(3-2)/3 = 1/3$). However, for the interval of 6, since its length is divisible by 3, the elements of the tuple fall into the same residue class modulo 3, thereby releasing an additional phase. The survival fraction increases to $(3-1)/3 = 2/3$. Thus, their local densities differ by exactly a finite factor ($k = 2$). They are rigidly connected within the deterministic "reducer" of the number line.
- **Generalization to $n$-Tuple Configurations of Arbitrary Form:** Arbitrary $n$-tuple primes of free configuration may possess different scaling coefficients $k$. However, the coefficient $k$ always remains strictly finite. Since any internal interval of the tuple decomposes into a finite number of prime factors, and the total number of elements in the $n$-tuple is finite, the resulting coefficient of the combinatorial alignment of sequences is always expressed by a finite number. Consequently, all such sets are asymptotically connected via finite coefficients.

# 4 The Method of Dynamic Sieves

Let us consider a regular set of pairs $A_k = \{k, k+2\}$. Imagine two number lines arranged vertically and vertically offset by two units relative to one another. For a set of $n$ elements, $n$ offset and aligned vertical straight lines are constructed. Consequently, each pair from the set $A_k$ assumes a strictly horizontal position. Imagine that these straight lines are rigidly linked into a single block and move synchronously downward past a horizontal and stationary detector-ray (Table 1). The ray instantaneously tests the passing tuples of numbers for divisibility based on a sequential basis of prime numbers $2, 3, 5, 7, \ldots$
When dividing by the number 7: out of 7 cases, the left number is divisible in only one case, and in the 6 other cases, it is not divisible. The right number is likewise not divisible in 6 out of 7 cases. However, since they cannot be divisible by 7 simultaneously due to the coprime shift, the prohibition of the right number falls upon one of the favorable cases of the left one. Out of 6 variants, the right number is not divisible in 5 cases. That is, in 5 out of 7 cases, both numbers are simultaneously not divisible by 7.

| | | |
|---|---|---|
| 13 | | 15 |
| 12 | | 14 |
| 11 | | 13 |
| 10 | | 12 |
| 9 | | 11 |
| 8 | | 10 |
| 7 | | 9 |
| 6 | | 8 |
| **5** | | **7** |
| 4 | | 6 |
| 3 | | 5 |

Table 1: Twin primes

| | | | | | | |
|---|---|---|---|---|---|---|
| 104 | | 106 | | 110 | | 112 |
| 103 | | 105 | | 109 | | 111 |
| 102 | | 104 | | 108 | | 110 |
| **101** | | **103** | | **107** | | **109** |

| | | | | | | |
|---|---|---|---|---|---|---|
| 15 | | 17 | | 21 | | 23 |
| 14 | | 16 | | 20 | | 22 |
| 13 | | 15 | | 19 | | 21 |
| 12 | | 14 | | 18 | | 20 |
| **11** | | **13** | | **17** | | **19** |
| 10 | | 12 | | 16 | | 18 |

Table 2: quadruplets

For a pair, the "survival" fraction through the filters of the ray equals:

$$\Pi_2 = \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{3}{5} \cdot \frac{5}{7} \cdot \frac{9}{11} \cdots = \frac{1}{2} \prod_{i=2}^{\infty} \frac{p_i - 2}{p_i} \tag{10}$$

For quadruplets (Table 2): when passing through the filters of prime numbers, one out of 5 four-tuples is sifted out through the modulus 5, 3 out of 7 four-tuples are sifted out through the modulus 7, and 7 out of 11 — through the modulus 11. The total flow is calculated by an infinite product:

$$\Pi_4 = \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{5} \cdot \frac{3}{7} \cdot \frac{7}{11} \cdot \frac{9}{13} \cdot \cdots = \frac{1}{6} \prod_{i=3}^{\infty} \frac{p_i - 4}{p_i} \tag{11}$$

A similar algorithm is applicable for higher dimensions as well. We are not interested in the calculations themselves. We are interested in the coefficient preceding the product sign. It will differ for different configurations of the same dimension (Tables 3–5), but the right-hand side equalizes all of them, regardless of their "status." This is like gears in a reducer: we do not need to count the number of teeth in the gears — it is sufficient to know the ratios of their diameters.

## 4.1 Invariance of Tail Structures and Projective Analysis of Tuples

By virtue of the inextricable unity of alternative and connected numerical sets, infinite products of residues describe the phase states of the holistic, connected system of the natural number sequence. It is necessary to emphasize that the subject of this study lies beyond the boundaries of the classical theory of prime numbers: the latter represent merely a special case ("a drop in the ocean") within the invisible ocean of alternative and connected sets. Examples operating with prime numbers are used exclusively as a representative demonstration of general regularities.

Let us examine the matching mechanism in greater detail using the example of two-dimensional prime tuples (twin pairs). The infinite products of densities for different configurations (intervals) differ exclusively in their finite "head" part, the numerator structure of which is determined by the geometry of the interval. Moreover, the length of the "head" part is invariant relative to the absolute magnitude of the interval between the elements of the pair: it is in a strict functional dependence solely on the prime divisors into which this magnitude decomposes.
For the mathematical formalization of this property, we apply the method of tabular reduction:

1. We construct an infinite system containing all potential finite configurations of prime pairs (or alternative elements).

2. We map each configuration to its characteristic infinite product of distribution density. Conceptually, we arrange these product chains into a structured infinite table where the rows are aligned strictly one below the other.

3. We observe that the asymptotic "tail" of the infinite product (excluding the initial finite segment) completely coincides for absolutely all rows of this table, regardless of the internal configuration of the tuple.

By performing the normalization operation — that is, by canceling the infinite product chains by their common, invariant "tail" component — we eliminate the divergent part of the process. As a result of this reduction, the infinite problem is reduced to the analysis of a matrix of individual finite coefficients.

| | | | | | | |
|---|---|---|---|---|---|---|
| 65 | | 73 | | 77 | | 85 |
| 63 | | 71 | | 75 | | 83 |
| 61 | | 69 | | 73 | | 81 |
| **59** | | **67** | | **71** | | **79** |
| 19 | | 27 | | 31 | | 37 |
| 17 | | 25 | | 29 | | 35 |
| 15 | | 23 | | 27 | | 33 |
| 13 | | 21 | | 25 | | 31 |
| **11** | | **19** | | **23** | | **29** |

Table 3: interval 8-4-8

| | | | | | | |
|---|---|---|---|---|---|---|
| **61** | | **67** | | **73** | | **79** |
| 45 | | 51 | | 57 | | 63 |
| 43 | | 49 | | 55 | | 61 |
| **41** | | **47** | | **53** | | **59** |
| 17 | | 23 | | 29 | | 35 |
| **11** | | **17** | | **23** | | **29** |
| 9 | | 15 | | 21 | | 27 |
| 7 | | 13 | | 19 | | 25 |
| **5** | | **11** | | **17** | | **23** |

Table 4: interval 6

| | | | | | | |
|---|---|---|---|---|---|---|
| **67** | | **97** | | **127** | | **157** |
| **41** | | **71** | | **101** | | **131** |
| 39 | | 69 | | 99 | | 129 |
| **37** | | **67** | | **97** | | **127** |
| 15 | | 45 | | 75 | | 105 |
| **13** | | **43** | | **73** | | **103** |
| **11** | | **41** | | **71** | | **101** |
| 9 | | 39 | | 69 | | 99 |
| **7** | | **37** | | **67** | | **97** |

Table 5: interval 30

# 5 The Global Theorem on Twin Primes

**Theorem 1.** *(V. P. Voevodov). For any ordered tuple of $n$ arbitrarily chosen prime numbers (or elements belonging to an alternative set with respect to the set of prime* *numbers) satisfying the condition $n < P_1 < P_2 < \cdots < P_n$, the fixed spacing template defined by them repeats within the sequence of natural numbers an infinite number of times.*

*Proof.* We apply the method of proof by contradiction. Suppose that for some fixed number of elements $n$, the given regular tuple has only a finite number of exact copies (twins) within the natural number sequence. By virtue of the finiteness of this subset, its asymptotic density equals zero. From this assumption, it follows that the asymptotic densities of all sets connected with it must also be strictly zero, since their local densities can exceed the initial density only by a finite factor ($k < \infty$).
However, this consequence is refuted by the following deterministic properties of the system:

1. According to Euclid's classical theorem, beyond any preassigned boundary, prime numbers continue to be distributed infinitely. From this infinite basis of elements, one can always constructively isolate subsets of two, three, or arbitrary $n$ prime numbers belonging to a specific connected set.

2. Since the densities of alternative and connected sets are invariantly related to one another via constant finite coefficients, the continuous, infinite process of element distribution within the basis inevitably, automatically, and with a fixed frequency fills the free phases of the connected templates, thereby forming exact geometric copies of the initial tuple.

Thus, the assumption that the number of the tuple's twins is finite completely destroys the rigid arithmetic connection between the infinite basis of prime numbers and the structure of connected sets uniquely determined by the number $n$.
The resulting contradiction refutes the initial assumption. Consequently, any $n$-tuple of free configuration satisfying the condition $n < P_1 < P_2 < \cdots < P_n$ repeats within the natural number sequence an infinite number of times. □

# 6 Conclusion

In this paper, a fundamentally new perspective on the foundations of structural arithmetic and the distribution of numerical sequences is presented. The developed concept of alternative and connected sets, governed by binding codes (BC), allows for overcoming the traditional "analytical barrier" of number theory. The main methodological conclusion of this work is that the study of prime number distribution cannot be conducted in isolation: they represent merely a local projection ("a drop in the ocean") of a global, deterministic reservoir of invariant numerical structures.
The method of dynamic sieves and the kinematic model of the number line reducer have enabled the transition of the highly complex problem of infinite product divergence into the domain of finite combinatorial analysis. The proven Global Theorem on $n$-Tuple Primes of Free Configuration not only generalizes classical conjectures on prime tuples

but also demonstrates that higher set-theoretic abstractions (in the spirit of G. Cantor's ideas) can assume entirely concrete, constructive arithmetic forms on the set of natural numbers. The author expresses his deepest gratitude to the Correspondence Youth Mathematical School (ZYuMSh) at Perm State University, which laid the foundational base of combinatorial and set-theoretic thinking that made this discovery possible.

## References

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