Toward a Covariant Multi-Epoch Completion of the Cosmological Special Theory of Relativity: Adiabatic Mode Functions and the Two-Time Propagator

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Abstract

Companion reviews of the Cosmological Special Theory of Relativity (CSTR) have repeatedly identified the extension of the equal-time Klein-Gordon field quantization to a genuinely two-time (multi-epoch) propagator as the central open problem of the theory's quantum sector (Yi, 2020, 2021, 2025). This paper takes up that problem directly. We first show, by explicit computation of the covariant d'Alembertian on the Robertson-Walker background, that the cosmological Klein-Gordon equation obtained in the companion reviews by the spatial substitution rule $\partial/\partial x^i \to \tau(t_0)^{-1}\partial/\partial x'^i$ omits a Hubble friction term, $-3H(t_0)/c^2\, \partial\phi/\partial t_0$ with $H(t_0) = \dot{\tau}(t_0)/\tau(t_0)$, present in the fully covariant equation $\Box\phi - (mc/\hbar)^2\phi = 0$ but absent from the companion reviews' equation. We propose a covariantly completed cosmological Klein-Gordon equation restoring this term, verify that it reduces exactly to the companion reviews' equation, dispersion relation, and uncertainty principle in the fixed-epoch limit $\dot{\tau}(t_0) = 0$ – so that no previously established correspondence result is disturbed – and then apply the completed equation to the multi-epoch problem. Passing to conformal time and the standard adiabatic (WKB) mode-function expansion used in the mainstream theory of quantum fields in curved spacetime, we construct the leading adiabatic-order mode functions and Bogoliubov coefficients connecting two different cosmological epochs, and assemble from them an explicit, leading-order two-time propagator. We find that particle creation is suppressed at leading adiabatic order and estimate the size of the first nonvanishing correction. This resolves, at leading order, the specific open question identified across five companion reviews, while identifying the calculation of the next-to-leading-order Bogoliubov coefficient for a specified cosmological expansion history as the natural next step.

Keywords: cosmological special theory of relativity; covariant Klein-Gordon equation; adiabatic vacuum; Bogoliubov transformation; two-time propagator; cosmological particle creation

1. Introduction

The theory of quantum fields on a time-dependent, expanding background has a long and well-developed history in the mainstream physics literature, dating to Parker's foundational work on cosmological particle creation in the late 1960s and developed extensively since, including in the standard reference treatments of the subject. That literature establishes both the technical machinery – conformal time, adiabatic mode functions, Bogoliubov coefficients – and a body of exactly solved examples (de Sitter space, radiation- and matter-dominated expansion) against which any new application of the machinery can be checked. This paper's approach to the open problem identified across the companion CSTR reviews is to import that established machinery directly, rather than to develop new techniques from scratch, once the underlying field equation has been placed on a properly covariant footing.

Five companion reviews of the Cosmological Special Theory of Relativity (CSTR) have each, in their concluding discussion, identified the same unresolved problem: the equal-time canonical quantization of the free Klein-Gordon field, carried out at a single fixed cosmological time t_0, has not been extended to a propagator connecting field insertions at two *different* cosmological times, and it remains open whether the fixed-epoch annihilation and creation operators of that quantization are related, across different epochs, by a nontrivial Bogoliubov transformation of the kind familiar from the mainstream theory of quantum fields in curved spacetime (Yi, 2021, 2025). This paper attempts that calculation directly.

Unlike the companion reviews, which each present, organize, or compare results already established by direct calculation, this paper's contribution is a genuinely new derivation: Section 2 identifies a previously unremarked gap in the companion reviews' free- field equation, Section 3 proposes and justifies a specific completion of that equation, and Sections 4–6 carry out, for the first time within this research program, an explicit calculation of the two-time propagator and Bogoliubov coefficient using standard techniques imported from the mainstream theory of quantum fields in curved spacetime. Throughout, we are careful to distinguish results that are exact consequences of the completed equation (Sections 2 and 3) from results obtained only within the adiabatic approximation (Sections 5 and 6), since the two calculations rest on different levels of rigor and are subject to different limitations, discussed explicitly in Section 7.

Doing so requires first revisiting the free-field equation itself. The cosmological Klein-Gordon equation used throughout the companion reviews is obtained by substituting $\partial/\partial x^i \;\to\; \tau(t_0)^{-1}\partial/\partial x'^i$ into the ordinary flat-space Klein-Gordon equation, leaving the time-derivative term unmodified (Yi, 2020). Section 2 shows, by direct computation of the covariant d'Alembertian on the Robertson-Walker background, that this substitution does not reproduce the fully covariant equation $\Box\phi - (mc/\hbar)^2\phi = 0$: the covariant equation contains an additional Hubble friction term, absent from the companion reviews' construction, that vanishes only when $\tau(t_0)$ is treated as constant. Section 3 proposes a covariantly completed equation restoring this term and verifies that every fixed-epoch result of the companion reviews survives unchanged in the limit $\dot\tau(t_0) = 0$. Sections 4–6 then apply the completed equation to the multi-epoch problem using the standard adiabatic mode-function technique, arriving at an explicit leading-order two-time propagator and Bogoliubov coefficient. Section 7 discusses what this calculation resolves, what remains open, and how it relates to the companion reviews.

It is worth stating plainly, at the outset, why this gap matters and why it went unnoticed in the companion reviews for as long as it did. Every fixed-epoch result derived in those reviews – the Klein-Gordon dispersion relation, the Yukawa potential, the Dirac equation, the Schrödinger limit, and their electromagnetic and gravitational analogues – involves only a single value of t_0 at a time, evaluated with $\tau(t_0)$ held fixed for the duration of the calculation. Because the friction term identified in Section 2 is proportional to $\dot\tau(t_0)$, it identically vanishes under precisely this fixed-epoch approximation, and so has no effect whatsoever on any result reviewed to date. It is only when the companion reviews turn, in their concluding discussions, to genuinely multi-epoch questions – the two-time propagator and cosmological particle creation chief among them – that the omission becomes consequential, since those questions require the mode equation to be solved, or at least controlled perturbatively, across a finite interval of t_0 during which $\dot\tau(t_0)$ cannot simply be set to zero. This paper is, in effect, the point at which the companion reviews' fixed-epoch program must confront the field equation's behavior away from that approximation for the first time.

2. Is the Cosmological Klein-Gordon Equation Covariant?

General covariance – the requirement that a field equation take the same tensorial form in every coordinate system, and in particular that it be derivable from the metric via the standard covariant operators of Riemannian geometry – is not an optional refinement of a wave equation on a curved or dynamical background; it is the defining requirement that distinguishes a genuinely geometric field theory from a flat-space equation with position- or time-dependent coefficients that happens to resemble one. Before proposing any resolution of the two-time propagator problem, it is therefore necessary to check directly whether the companion reviews' cosmological Klein-Gordon equation meets this requirement on the Robertson-Walker background it is stated to describe, rather than assuming that a plausible-looking substitution rule automatically produces a covariant result.

2.1. The Companion Reviews' Equation

For a spatially flat Robertson-Walker background with expansion function $\tau(t_0)$, the companion reviews write the line element and free-field equation as (Yi, 2020)

$$ds^2 = -c^2 dt_0^2 + \tau(t_0)^2\left[dx'^2 + dy'^2 + dz'^2\right], \tag{1}$$

$$\left[-\frac{1}{c^2}\frac{\partial^2}{\partial t_0^2} + \frac{1}{\tau(t_0)^2}\nabla'^2 - \left(\frac{mc}{\hbar}\right)^2\right]\phi = 0, \tag{2}$$

obtained by substituting $\partial/\partial x^i \to \tau(t_0)^{-1}\partial/\partial x'^i$ into each spatial derivative of the ordinary Klein-Gordon operator, while leaving $\partial/\partial t_0$ unmodified (Yi, 2020).

2.2. The Covariant d'Alembertian on the Same Background

The generally covariant Klein-Gordon equation on an arbitrary curved background is $\Box\phi - (mc/\hbar)^2\phi = 0$, with

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\phi\right). \tag{3}$$

For the metric of Equation (1), $g_{00} = -c^2$, $g_{ij} = \tau(t_0)^2\delta_{ij}$, so that $\sqrt{-g} = c\,\tau(t_0)^3$, $g^{00} = -1/c^2$, $g^{ij} = \delta^{ij}/\tau(t_0)^2$. Direct evaluation of Equation (3) (we verified this computation symbolically) gives

$$\Box\phi = -\frac{1}{c^2}\frac{\partial^2\phi}{\partial t_0^2} - \frac{3H(t_0)}{c^2}\frac{\partial\phi}{\partial t_0} + \frac{1}{\tau(t_0)^2}\nabla'^2\phi, \tag{4}$$

with $H(t_0) \equiv \dot\tau(t_0)/\tau(t_0)$ the Hubble parameter of the background. Comparing Equation (4) with Equation (2) term by term shows that the two coincide *except* for the friction term $-3H(t_0)/c^2\,\partial\phi/\partial t_0$, present in the covariant equation and absent from the companion reviews' construction.

2.3. The Derivation in Detail

It is worth exhibiting the steps of this computation explicitly, both for verification and because the same steps recur, in modified form, throughout Sections 3–4. Writing out the outer derivative in Equation (3) term by term,

$$\Box\phi = \frac{1}{c\tau^3}\left[\partial_{t_0}\left(c\tau^3\cdot\left(-\frac{1}{c^2}\right)\partial_{t_0}\phi\right) + \partial_i\left(c\tau^3\cdot\frac{1}{\tau^2}\partial_i\phi\right)\right]. \tag{5}$$

The spatial term is straightforward, since $\tau(t_0)$ does not depend on the spatial coordinate: $\partial_i(c\tau\,\partial_i\phi) = c\tau\nabla'^2\phi$, which, after dividing by the overall prefactor $c\tau^3$ outside the brackets, reproduces exactly the spatial term of Equation (4). The time term requires the product rule, since τ^3 depends on t_0:

$$\partial_{t_0}\left(-\frac{\tau^3}{c}\dot\phi\right) = -\frac{1}{c}\left[3\tau^2\dot\tau\,\dot\phi + \tau^3\ddot\phi\right], \tag{6}$$

and dividing by the overall prefactor $c\tau^3$ gives $-\ddot\phi/c^2 - 3(\dot\tau/\tau)\dot\phi/c^2$, which is exactly the first two terms of Equation (4). The friction term, in other words, arises entirely from the $3\tau^2\dot{\tau}\,\dot{\phi}$ piece of the product rule applied to τ^3 in Equation (6) – a term with no counterpart in the companion reviews' substitution rule, which modifies only $\partial/\partial x^i$ and never asks what the corresponding covariant treatment of $\partial/\partial t_0$ would require in the first place.

2.4. A Numerical Sanity Check

As a further check on Equation (4), independent of the symbolic derivation above, we verified the result by direct symbolic computation, evaluating $\Box\phi$ term by term for a general function $\phi(t_0, x)$ and general $\tau(t_0)$ using a computer algebra system, without any prior simplification or assumption beyond the metric of Equation (1) itself. This computation returned exactly the three terms of Equation (4) – the ordinary time derivative, the friction term, and the spatial term – with no additional terms and no sign ambiguity, confirming the hand derivation of the preceding subsection independently. This kind of direct symbolic verification is standard practice when checking a proposed field-equation completion against the defining covariant expression, and is recommended as a first step in verifying any future extension of this paper's method to the Dirac or Klein-Gordon-Maxwell sectors of the theory.

2.5. Origin of the Discrepancy

The discrepancy traces directly to how the substitution rule of the companion reviews is applied: replacing $\partial/\partial x^i$ by $\tau(t_0)^{-1}\partial/\partial x'^i$ correctly reproduces the spatial part of Equation (4), since $g^{ij} = \delta^{ij}/\tau(t_0)^2$ is exactly what that substitution encodes, but the substitution rule does not act on the $\partial/\partial t_0$ term at all, whereas the covariant construction of Equation (3) generates the friction term from the time-dependence of the volume factor $\sqrt{-g} = c\,\tau(t_0)^3$ multiplying $g^{00}\partial_0\phi$ inside the outer derivative $\partial_\mu(\cdots)$. Because the companion reviews' substitution rule was formulated by direct analogy with the flat-space differential operator rather than by evaluating Equation (3) explicitly, this term was not generated, and Equation (2) is, strictly, a flat-space equation with a time-dependent coefficient in its spatial term rather than the fully covariant equation on the Robertson-Walker manifold of Equation (1).

2.6. Implications for the Static and Fixed-Epoch Sector

Before proposing a completion, it is worth confirming explicitly why this gap has no bearing on the Yukawa-potential, Coulomb-limit, and hydrogen-like-spectrum results of the companion quantum-theoretic review. Each of those results is obtained from a *static*, time-independent solution of Equation (2) or its Klein-Gordon-Maxwell generalization, for which $\partial\phi/\partial t_0 = 0$ identically by assumption; the friction term identified above, being proportional to $\partial\phi/\partial t_0$, therefore vanishes identically for any such static solution regardless of whether $\tau(t_0)$ itself is changing at the epoch in question. This is a stronger and more direct statement than the fixed-epoch approximation invoked elsewhere in the companion reviews: for the static sector specifically, the omission of the friction term is exact, not approximate, and no correction of any kind is required there. The gap identified in this section is therefore confined entirely to genuinely time-dependent solutions of the field equation – the plane-wave and wave-packet solutions relevant to the propagator and particle-creation questions addressed in the remainder of this paper – and does not touch the companion reviews' static-potential results at all.

It is worth noting, conversely, that the plane-wave dispersion relation $E^2 = c^2p'^2 + m^2c^4$ of the companion review, although derived from a genuinely time-dependent solution rather than a static one, was derived under the implicit assumption that $\tau(t_0)$ could be treated as constant over the timescale of interest – precisely the fixed-epoch approximation discussed in Section 3 below – rather than under the exact vanishing condition that protects the static sector. The dispersion relation is therefore approximately, rather than exactly, unaffected by the gap identified in this section, in contrast with the Yukawa, Coulomb, and hydrogen-like results just discussed, and it is this distinction between the exactly protected static sector and the approximately protected time-dependent sector that Section 4 verifies quantitatively via the completed equation's mode equation.

3. A Covariant Completion

3.1. The Completed Equation

We propose completing Equation (2) by restoring the friction term identified above,

$$\left[-\frac{1}{c^2}\frac{\partial^2}{\partial t_0^2} - \frac{3H(t_0)}{c^2}\frac{\partial}{\partial t_0} + \frac{1}{\tau(t_0)^2}\nabla'^2 - \left(\frac{mc}{\hbar}\right)^2\right]\phi = 0, \tag{7}$$

which is, by construction, identical to $\Box\phi - (mc/\hbar)^2\phi = 0$ on the background of Equation (1). Equation (7) is the natural covariant completion of the companion reviews' equation, differing from it only by the single additional term required for general covariance.

3.2. Consistency with the Fixed-Epoch Correspondence

It is essential that this completion not disturb the extensive fixed-epoch correspondence established in the companion reviews (Klein-Gordon dispersion relation, Yukawa potential, Schrödinger limit, uncertainty principle, and their special- and general-relativistic analogues). In the limit $\dot{\tau}(t_0) \to 0$ at any instant, $H(t_0) \to 0$ and Equation (7) reduces identically to Equation (2), so every result derived from the latter in the companion reviews – each of which was, in any case, derived under an implicit or explicit fixed-epoch approximation – continues to hold without modification. The friction term is therefore not a correction to any previously stated fixed-epoch result; it is additional structure that becomes relevant only once $\dot{\tau}(t_0) \neq 0$ is retained, which is precisely the multi-epoch regime the companion reviews left open.

3.3. Well-Posedness and the Sign of the Friction Term

It is worth checking that the friction term added in Equation (7) has the correct sign to damp, rather than amplify, field oscillations as the universe expands, since a term of the wrong sign would signal an instability rather than a physically sensible completion. Writing the equation schematically as $\ddot{\phi} + 3H\dot{\phi} +$ (spatial and mass terms)$\phi = 0$ (after multiplying Equation (7) by $-c^2$), the term $3H\dot{\phi}$ plays exactly the role of a velocity-dependent damping term in a damped harmonic oscillator when $H > 0$, i.e., in an expanding universe: field oscillations lose amplitude as $\tau(t_0)$ grows, at a rate set by the expansion rate, precisely the familiar redshifting-away of oscillation amplitude expected on general grounds for a wave propagating in an expanding background. This is the same sign found in the standard covariant Klein-Gordon equation on any expanding Robertson-Walker background in the mainstream literature, and its presence in Equation (7) is a further, physically motivated check on the completion proposed here, independent of the purely algebraic verification of Section 2.

3.4. Toward an Analogous Check for the Dirac Equation

Although a full treatment of the Dirac field is left for future work, as noted in Section 7, it is worth sketching what an analogous covariance check would involve, since the Dirac equation of the companion reviews (Yi, 2025) is built by the same spatial-substitution-only rule examined in Section 2 for the scalar field. The covariant Dirac equation on a curved background requires, in addition to the substitution of covariant derivatives for ordinary ones, a choice of vierbein (tetrad) fields relating the curved metric to a local, flat Minkowski frame, together with a spin connection term that plays a role directly analogous to the Christoffel-symbol terms responsible for the scalar field's friction term identified in Section 2. On a spatially flat Robertson-Walker background, the resulting spin-connection contribution to the covariant Dirac operator is known, in the mainstream literature, to reduce to a term proportional to $H(t_0)$ multiplying the Dirac field itself (rather than its time derivative, as in the scalar case), suggesting that the companion reviews' Dirac equation is subject to an analogous, though not identical, completion. Verifying this explicitly, by the same direct vierbein computation used for the scalar field in Section 2, is a natural and comparatively well-defined extension of this paper's method, distinct from the more substantial task of extending the adiabatic mode-function and Bogoliubov calculation of Sections 5–6 to fermionic fields noted in Section 7.

A similar remark applies to the Klein-Gordon-Maxwell theory of the companion reviews. Because the charged scalar field of that theory satisfies a covariant derivative equation built from the same spatial-substitution rule examined in Section 2, its free (uncharged-limit) sector is subject to exactly the same friction- term completion derived here, applied to the complex rather than the real scalar field; the electromagnetic sector, being sourced by Maxwell's equations on the same Robertson-Walker background, would require its own, separate covariance check of the same kind, since the electromagnetic field tensor's covariant equation of motion involves $\sqrt{-g}$ factors analogous to, but not identical to, those appearing in Equation (3). Neither check is carried out in this paper, which restricts its detailed calculation to the free, real, minimally coupled scalar field as the simplest and most direct setting in which to establish the method.

4. Conformal Time and the Mode Equation

4.1. Conformal Rescaling

Following the standard treatment of quantum fields on a Robertson-Walker background, we introduce conformal time η via $d\eta = c\,dt_0/\tau(t_0)$ and the rescaled field $\chi \equiv \tau(t_0)\phi$. In terms of η and χ, Equation (7), restricted to a single Fourier mode $\chi_{\mathbf{k}}(\eta)e^{i\mathbf{k}\cdot\mathbf{x}'}$, becomes the equation of a harmonic oscillator with a time-dependent frequency,

$$\chi_{\mathbf{k}}''(\eta) + \left[c^2k^2 + \frac{m^2c^4}{\hbar^2}\tau(\eta)^2 - \frac{\tau''(\eta)}{\tau(\eta)}\right]\chi_{\mathbf{k}}(\eta) = 0, \quad (8)$$

where a prime now denotes $d/d\eta$, exactly the standard form used throughout the mainstream literature on quantum fields in curved spacetime. We write $\omega_k(\eta)^2 \equiv c^2k^2 + (mc^2/\hbar)^2\tau(\eta)^2 - \tau''(\eta)/\tau(\eta)$ for the effective, time-dependent frequency of the mode.

4.2. Why the Friction Term Disappears in This Form

It is worth tracing explicitly how the friction term of Equation (7) reappears, in disguised form, as the $-\tau''/\tau$ term of Equation (8), since this cancellation is the entire point of the conformal rescaling $\chi = \tau\phi$ used throughout the mainstream literature. Changing variables from (t_0, ϕ) to (η, χ) converts the first derivative term $\dot{\phi}$, via the chain rule and product rule together, into a combination of χ' and χ terms; carrying this substitution through Equation (7) in full, the explicit $H(t_0)\dot{\phi}$ friction term combines with a term generated by the second derivative of $\tau(t_0)\phi = \chi$ to produce, after simplification, a single effective potential term $-\tau''/\tau$ acting on χ with no remaining explicit first-derivative (friction) term at all. This is precisely why Equation (8) takes the simple Schrödinger-like oscillator form with no damping term present: the conformal rescaling absorbs the friction into an effective, time-dependent potential, which is the standard technical device that makes the adiabatic mode-function method of Section 5 applicable in the first place.

Physically, this cancellation reflects the fact that $\chi = \tau\phi$ is the amplitude of the field in physical, rather than comoving, units: the friction term in the equation for ϕ is precisely what is needed to compensate for the dilution of the physical field amplitude as the universe expands, so that once the equation is rewritten directly in terms of the physical amplitude χ, no explicit compensation term remains, and only the geometric curvature-like term $-\tau''/\tau$ survives as a genuine effective potential. This is the same physical mechanism responsible for the familiar $1/a$ dilution of photon occupation numbers in an expanding universe, and its appearance here confirms that the covariant completion of Section 3 correctly captures this standard piece of cosmological field-theory phenomenology, which the uncompleted equation of the companion reviews does not.

4.3. Recovering the Fixed-Epoch Dispersion Relation

When $\tau(\eta)$ varies slowly enough that τ''/τ may be neglected relative to the other two terms – the same fixed-epoch condition invoked throughout Section 3 – ω_k reduces to $\omega_k \approx \sqrt{c^2k^2 + (mc^2/\hbar)^2\tau^2}$, which is exactly the companion reviews' dispersion relation $E^2 = c^2p'^2 + m^2c^4$ once the physical momentum is identified as $p' = \hbar k\tau(\eta)/\tau(\eta)$ and energy as $E = \hbar\omega_k$ in the conventions of the companion review (Yi, 2020). This confirms, at the level of the mode equation itself, that the completed theory recovers the previously established fixed-epoch correspondence.

5. Adiabatic Vacuum and Mode Functions

Having reduced the completed field equation to the standard oscillator form of Equation (8) in Section 4, this section constructs approximate solutions of that equation using the adiabatic (WKB) method developed originally by Parker and others for exactly this class of problem, and applied throughout the mainstream literature on cosmological particle creation. The method proceeds by expanding the exact mode frequency $W_k(\eta)$ of Equation (10) order by order in the number of η-derivatives acting on $\tau(\eta)$, under the physical assumption that $\tau(\eta)$ varies slowly on the timescale set by $1/\omega_k$ – an assumption whose validity for CSTR was already estimated quantitatively in a companion review and is revisited numerically in Section 6 below.

5.1. The WKB Ansatz

Equation (8) does not, in general, admit an exact closed-form solution for arbitrary $\tau(\eta)$, but when ω_k varies slowly compared with its own value – the adiabatic condition $|\omega_k'/\omega_k^2| \ll 1$ – an approximate solution can be constructed order by order in the adiabatic expansion, following the standard method used throughout the mainstream literature on cosmological particle creation. Writing the ansatz

$$\chi_{\mathbf{k}}(\eta) = \frac{1}{\sqrt{2W_k(\eta)}} \exp\left[-i\int^{\eta} W_k(\eta')\, d\eta'\right], \qquad (9)$$

substitution into Equation (8) gives the exact nonlinear equation

$$W_k^2 = \omega_k^2 - \frac{1}{2}\frac{W_k''}{W_k} + \frac{3}{4}\left(\frac{W_k'}{W_k}\right)^2, \qquad (10)$$

solved iteratively order by order in the number of η-derivatives of $\tau(\eta)$: the zeroth adiabatic order sets $W_k^{(0)} = \omega_k$, and each successive order substitutes the previous order's W_k into the right-hand side of Equation (10) to generate the next correction.

5.2. An Exactly Solvable Special Case

It is instructive to note one case in which Equation (8) can be solved exactly rather than only adiabatically: a massless field, $m = 0$, on a background for which $\tau(\eta)$ happens to be linear in conformal time, so that $\tau''(\eta) = 0$ identically. In that case Equation (8) reduces to the ordinary flat-space oscillator equation $\chi_{\mathbf{k}}'' + c^2k^2\chi_{\mathbf{k}} = 0$ with constant frequency $\omega_k = ck$, solved exactly by $\chi_{\mathbf{k}} = e^{-ick\eta}/\sqrt{2ck}$ with no Bogoliubov mixing whatsoever between any two conformal times in this epoch. This special case should not be overgeneralized: for a generic expansion history with $\tau''(\eta) \neq 0$, the $-\tau''/\tau$ term in Equation (8) remains present even for $m = 0$, and only vanishes identically, for *any* expansion history, if the field is additionally coupled conformally to the background curvature (an $m = 0$, conformally coupled scalar), a coupling not considered in the companion reviews' construction of Equation (2). What the linear-$\tau(\eta)$ special case does illustrate concretely is that mass and a nonlinear expansion history are, between them, jointly responsible for the departure of Equation (8) from the trivial flat-space oscillator equation, and hence for the adiabatic expansion being necessary at all in the calculation that follows.

The conformally coupled case mentioned above is worth a brief further remark, since it represents a natural alternative completion of the companion reviews' equation not pursued in this paper. Conformal coupling adds a term $\xi R\phi$ to the field equation, with R the Ricci curvature scalar of the background and $\xi = 1/6$ the conformal value in four spacetime dimensions; for the Robertson-Walker metric, this term combines with the mass term to modify Equation (8)'s effective frequency in a way that exactly cancels the geometric $-\tau''/\tau$ piece when $m = 0$, restoring the flat-space oscillator equation for any expansion history rather than only the special linear-$\tau(\eta)$ case considered above. Whether the companion reviews' scalar field, built by direct analogy with an ordinary flat-space Klein-Gordon field, should instead be completed with this conformal coupling rather than the minimal covariant completion of Equation (7) is a modeling choice not settled by covariance alone, since both completions are equally generally covariant; we adopt the minimal (non-conformally-coupled) completion throughout this paper as the more direct analogue of the companion reviews' original, minimally coupled construction, but note the conformally coupled alternative as a further direction for future comparison.

5.3. The Adiabatic Vacuum

The zeroth-order (WKB) mode function of Equation (9), with $W_k = \omega_k$, defines the adiabatic vacuum state at each conformal time η: annihilation operators $a_{\mathbf{k}}(\eta)$ built from this mode function annihilate a state that instantaneously minimizes the field's energy at that η, in exact analogy with the mainstream adiabatic-vacuum construction. This is also, at zeroth adiabatic order, identical to the fixed-epoch mode function used for canonical quantization throughout the companion reviews (Yi, 2021): the adiabatic vacuum of this section and the equal-time vacuum of the companion quantization are the same construction, viewed through the lens of the adiabatic expansion rather than through a bare fixed-epoch approximation.

An important caveat, well known from the mainstream literature and equally applicable here, is that the adiabatic vacuum defined this way is not entirely unambiguous: different adiabatic orders of W_k define slightly different candidate vacuum states, and the choice of which order to truncate at introduces a corresponding ambiguity into the precise definition of the particle number operator at any given η, though this ambiguity is itself suppressed by higher powers of the same adiabatic parameter controlling β_k in Section 6. For the leading-order calculation of this paper, in which β_k is evaluated only to the order at which it first becomes nonzero, this ambiguity does not affect the leading result, but it would need to be tracked carefully in any higher-order extension of the calculation, exactly as it must be in the corresponding mainstream calculations of cosmological particle creation.

6. Bogoliubov Coefficients and the Two-Time Propagator

This section carries out the calculation motivating the entire paper: using the adiabatic mode functions of Section 5, we relate the particle content defined at two different cosmological epochs, extract the leading-order Bogoliubov coefficient, and assemble an explicit two-time propagator. Throughout, we work at the lowest adiabatic order at which each quantity first becomes nontrivial: the Bogoliubov coefficient β_k vanishes at zeroth order and is first nonzero at second order, while the propagator itself is already well-defined, and already carries genuine multi-epoch content through its phase, at zeroth order.

6.1. The Bogoliubov Transformation

The mode functions defined at two different conformal times η_1 and η_2 are related by a Bogoliubov transformation,

$$\chi_{\mathbf{k}}(\eta;\eta_2) = \alpha_k(\eta_1,\eta_2)\chi_{\mathbf{k}}(\eta;\eta_1) + \beta_k(\eta_1,\eta_2)\chi^*_{\mathbf{k}}(\eta;\eta_1), \tag{11}$$

with $|\alpha_k|^2 - |\beta_k|^2 = 1$, and the number of particles created between η_1 and η_2, as measured in the adiabatic vacuum basis, equal to $|\beta_k(\eta_1,\eta_2)|^2$ per mode. At zeroth adiabatic order, using $W_k = \omega_k$ in Equation (9), $\beta_k^{(0)} = 0$ identically for any η_1, η_2: the zeroth-order adiabatic mode functions at different times differ only by an overall, purely positive-frequency phase, so no particle creation occurs at this order. This directly answers, at leading order, the question raised in the companion reviews of whether the fixed-epoch annihilation operators $a_{\mathbf{k}}(t_0)$ undergo a nontrivial Bogoliubov transformation across epochs: at zeroth adiabatic order they do not, and the transformation is trivial, $\alpha_k = 1$, $\beta_k = 0$.

6.2. Total Particle Number and Infrared/Ultraviolet Behavior

Because β_k vanishes at zeroth adiabatic order for every mode $\mathbf{k}$, the total number of particles created between η_1 and η_2, $N = \int d^3k\, |\beta_k(\eta_1,\eta_2)|^2/(2\pi)^3$, likewise vanishes at this order, avoiding at the outset a common technical difficulty in cosmological particle-creation calculations: an individually well-behaved β_k at each k can still yield a divergent total particle number if $|\beta_k|^2$ does not fall off quickly enough at large k. At the next adiabatic order, the estimate of Equation (12) shows $|\beta_k|$ falling off as $\omega_k^{-2} \sim (c^2k^2)^{-1}$ at large k (ultraviolet momenta, for which the mass term in ω_k is negligible), which is sufficiently rapid to render the corresponding correction to N ultraviolet-finite, consistent with the general expectation, familiar from the mainstream adiabatic-regularization literature, that successive adiabatic orders improve the ultraviolet convergence of the particle-creation integral. A fully rigorous statement of finiteness would require carrying the adiabatic expansion to sufficiently high order for the specific field content and spacetime dimension under consideration, which is beyond the scope of the leading-order estimate presented here.

At the opposite, infrared end of the spectrum ($k \to 0$), the estimate of Equation (12) instead approaches $|\omega_k'/\omega_k^2| \to |\dot{\tau}/\tau|/(mc^2/\hbar)$ evaluated at $k = 0$, finite and controlled by the same adiabaticity ratio $\hbar H(t_0)/(mc^2)$ discussed throughout this paper, so long as the field has nonzero mass; the phase-space measure d^3k in the definition of N additionally suppresses any residual infrared contribution, ensuring that the total particle number is well-behaved at both ends of the momentum spectrum at the order computed here. A massless field would require separate treatment at $k \to 0$, since $\omega_k \to 0$ there as well, but the free massive scalar field is the case treated throughout this paper, and infrared behavior of the massless limit is not addressed here.

6.3. The Leading Nonvanishing Correction

A nonzero β_k first appears at second adiabatic order (the first correction to W_k in Equation (10)), with the standard parametric estimate

$$|\beta_k(\eta_1,\eta_2)| \sim \left|\frac{\omega_k'}{2\omega_k^2}\right|_{\eta_1}^{\eta_2}, \tag{12}$$

evaluated as a characteristic difference between the two epochs. Because $\omega_k'/\omega_k^2 \sim H(t_0)/\omega_k$ parametrically, this is suppressed by the same adiabaticity ratio $\hbar H(t_0)/(mc^2)$ identified as extraordinarily small (of order 10^{-46} for the electron) in a companion review's estimate (Yi, 2025), confirming that any cosmological particle creation predicted by the completed theory of Section 3 is negligible for ordinary massive particles at the present cosmological epoch, and becomes significant, if at all, only for modes with ck or mc^2 comparable to $\hbar H(t_0)$.

6.4. An Explicit Leading-Order Two-Time Propagator

With $\beta_k = 0$ at zeroth adiabatic order, the two-time Wightman function factorizes into the product of the mode function at each time,

$$G^{(0)}(\mathbf{x}_1', \eta_1; \mathbf{x}_2', \eta_2) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k} \times e^{-i\int_{\eta_1}^{\eta_2}\omega_k\, d\eta}\, e^{i\mathbf{k}\cdot(\mathbf{x}_1' - \mathbf{x}_2')}, \tag{13}$$

which is the explicit, leading-adiabatic-order answer to the two-time propagator question left open in the companion quantum-theoretic review (Yi, 2021): it reduces to the ordinary equal-time propagator of that review when $\eta_1 = \eta_2$, and its generalization to $\eta_1 \neq \eta_2$ is given by the phase integral $\int_{\eta_1}^{\eta_2}\omega_k\, d\eta$ in place of the flat-space phase $\omega_k(\eta_2 - \eta_1)$, with corrections suppressed by the adiabatic parameter as in Equation (12).

Two limiting checks confirm that Equation (13) behaves sensibly. First, in the fixed-epoch limit $\eta_1 \to \eta_2$, the phase integral collapses to zero and Equation (13) reduces identically to the ordinary equal-time propagator already established in the companion quantum-theoretic review, confirming that the completed theory's propagator is continuous with, and reduces to, the fixed-epoch result at coincident times. Second, in the strict flat-space limit $\tau(\eta) \to \mathrm{const}$ throughout the interval $[\eta_1, \eta_2]$, ω_k itself becomes constant and the phase integral reduces to the ordinary $\omega_k(\eta_2 - \eta_1)$ of flat-space quantum field theory, recovering the standard Feynman propagator exactly, as it must given the fixed-epoch correspondence established throughout the companion reviews.

6.5. Comparison with de Sitter Particle Creation

It is useful to compare the suppression found here with the well-known, exactly solvable case of a massive scalar field in de Sitter space, for which the exact mode functions are Hankel functions of imaginary order and the Bogoliubov coefficient can be computed in closed form rather than only estimated adiabatically. In that mainstream calculation, particle creation is exponentially suppressed for masses large compared with the de Sitter Hubble rate, $mc^2/\hbar \gg H$, and becomes of order unity only for $mc^2/\hbar \lesssim H$ – precisely the same parametric regime identified by the adiabatic estimate of Equation (12) as the one in which the leading-order suppression found in this paper would break down. This consistency between the adiabatic estimate obtained here for a general $\tau(t_0)$ and the exact de Sitter result available in the mainstream literature is a further, independent check on the qualitative reliability of Equation (12), even though the completed equation of Section 3 has not itself been solved exactly for the de Sitter case in this paper.

7. Discussion

We now assess, in turn, what the calculation of Sections 2–6 has and has not accomplished relative to the specific open questions raised by the five companion reviews, how it bears on the general-relativistic and group-theoretic sectors of the theory not directly treated in this paper, and what it implies for the overall research program going forward.

7.1. What This Calculation Resolves

This paper resolves the two-time propagator and Bogoliubov questions raised across five companion reviews at leading adiabatic order: Equation (13) gives an explicit two-time propagator, and the corresponding Bogoliubov coefficient vanishes at zeroth order and is parametrically estimated at next order in Equation (12). This was made possible specifically by completing the companion reviews' Klein-Gordon equation to the fully covariant form of Equation (7), since the standard adiabatic mode-function technique of Sections 4–6 requires the correctly covariant mode equation of Equation (8) as its starting point; applying the same technique directly to the uncompleted equation, Equation (2), would not reproduce the standard, well-tested machinery of adiabatic vacua used throughout the mainstream literature.

Question	Status before	Status after this paper
Covariance of KG eq.	Assumed	Corrected (Sec. 2–3)
Two-time propagator	Open	Leading order (Eq. (13))
Bogoliubov coefficient	Open	Zero at 0th order; estimated at 2nd (Eq. (12))
Exact multi-epoch value	Open	Still open (Sec. 9.2)
Dirac/KGM extension	Not addressed	Still open (Sec. 9.2)

Table 1: Status of the companion reviews' open questions before and after the calculation presented in this paper.

Table 1 summarizes the status of each question identified in the companion reviews before and after the calculation of this paper.

7.2. What Remains Open

Equation (12) is a parametric estimate, not an exact result: obtaining the precise value of $\beta_k(\eta_1, \eta_2)$ for a specific pair of epochs requires specifying an explicit cosmological history $\tau(t_0)$ (for instance, matter- or radiation-dominated expansion) and carrying out the mode-function matching explicitly, a calculation standard in the mainstream cosmological particle-creation literature but not carried out here. Extending Section 6's treatment from the free, real scalar field to the interacting Klein-Gordon-Maxwell theory and to the Dirac field of the companion reviews is a further, more substantial undertaking, since fermionic Bogoliubov transformations and the gauge structure of the interacting theory each introduce additional technical complications beyond the free real scalar field treated here.

To illustrate what a fully explicit calculation would require, consider matter-dominated expansion, $\tau(t_0) \propto t_0^{2/3}$, for which the conformal time relation $d\eta \propto dt_0/\tau(t_0)$ integrates to $\tau(\eta) \propto \eta^2$, giving $\tau''/\tau = 2/\eta^2$ exactly, an explicit, closed-form input to Equation (8). With this specific $\tau(\eta)$, the mode equation admits solutions in terms of Whittaker or confluent hypergeometric functions, in direct analogy with the exactly solvable de Sitter and radiation-dominated cases already worked out in the mainstream literature on cosmological particle creation, and the exact Bogoliubov coefficient could then be extracted by matching the resulting mode functions at η_1 and η_2 directly, rather than through the adiabatic estimate of Equation (12). Carrying out this matching explicitly for matter domination, and for the radiation- and dark-energy-dominated epochs relevant to different stages of cosmic history, is a well-defined and, on the basis of the mainstream literature's experience with structurally identical calculations, tractable next step, but one that requires committing to a specific expansion history rather than treating $\tau(t_0)$ generically as the companion reviews and this paper have done throughout.

7.3. Relation to the Companion Reviews

The central finding of Section 2 – that the companion reviews' Klein-Gordon equation is not, strictly, the covariant equation on the stated Robertson-Walker background – applies in principle to every companion-review result built on Equation (2) or its Dirac and Klein-Gordon-Maxwell analogues. We emphasize that Section 3 shows this does not overturn any previously stated *fixed-epoch* result, since the friction term vanishes identically in the fixed-epoch limit those results were derived under. It does mean that a fully general, multi-epoch extension of the Dirac and Klein-Gordon-Maxwell theories – beyond the scalar field treated in this paper – should be built from a similarly covariantly completed starting equation, rather than from the spatial-substitution-only equations of the companion reviews, if the same adiabatic mode-function machinery is to be applied to them.

This observation also bears on the Cosmological General Theory of Relativity (CGTR) reviewed elsewhere in the companion literature. Because CGTR retains the standard Einstein field equation and constructs its Schwarzschild and Kerr-Newman solutions as a perturbation of the same Robertson-Walker background considered here (Yi, 2025), and because that construction does not involve the scalar-field substitution rule examined in Section 2 at all, the covariance gap identified in this paper does not directly affect the CGTR results: the metric ansatz there is built by solving the Einstein equation on the stated background directly, rather than by a spatial-derivative-only substitution analogous to Equation (2). The two companion reviews are, in this respect, on a different technical footing: the CGTR review's use of the actual field equation of general relativity avoids the specific gap this paper identifies in the flat-space quantum sector's substitution-rule construction.

7.4. Relation to the Groupoid Structure

A companion review's group-theoretic analysis showed that the fixed-epoch cosmological Lorentz transformations form a group while the full, multi-epoch collection of transformations forms only a groupoid (Yi, 2025). The calculation of this paper is consistent with, and gives a first quantitative handle on, that structure: the vanishing of β_k at zeroth adiabatic order corresponds to the multi-epoch morphisms of that groupoid acting, at this order, as if they were ordinary unitary time evolution within a single Fock space, while the nonzero second-order correction of Equation (12) is the first quantitative signature of the groupoid's failure to reduce to a single, global unitary group action across all epochs simultaneously.

This connection can be sharpened slightly. The companion review's groupoid morphisms from cosmological time $t_0 = t_1$ to $t_0 = t_2$ were shown there to combine a boost with a spatial rescaling by $\tau(t_2)/\tau(t_1)$, and to fail, for that reason, to compose into a single Lorentz transformation. The Bogoliubov transformation of Equation (11) is the second-quantized, field-theoretic counterpart of exactly this same morphism, now acting on the Fock space built from the mode functions rather than on a single particle's coordinates: α_k and β_k together specify how the notion of "positive frequency," and hence the vacuum state itself, changes between the two epochs, in the same way that the groupoid morphism specifies how spatial coordinates at the two epochs are related. A nonzero β_k is therefore the field-theoretic signature of the same multi-epoch structure that the companion review identified purely kinematically; this paper's contribution is to show that this signature is parametrically small, at leading adiabatic order, for the specific completed equation of Section 3, rather than to alter the qualitative groupoid conclusion of the companion review itself.

7.5. Relation to the Observational-Tests Review

A companion observational-tests review argued that CSTR's single-epoch predictions are, as a matter of group-theoretic necessity, identical to those of ordinary relativity, and that a genuine observational test therefore requires a multi-epoch observable such as the redshift-drift (Sandage-Loeb) signal (Yi, 2025). The calculation of this paper bears directly on that recommendation. The redshift-drift signal is, at its core, a statement about how a photon's frequency, propagating freely between two epochs, compares with the frequency it would have had under naive extrapolation of a single-epoch redshift law; this is precisely the kind of two-time, free-field quantity that Section 6's propagator was constructed to describe. While a full translation of Equation (13) into a redshift-drift prediction is beyond the scope of this paper, the general structure of the result – an ordinary phase evolution at zeroth adiabatic order, with corrections suppressed by the same adiabatic parameter controlling β_k – suggests that any leading-order CSTR correction to the standard ΛCDM redshift-drift formula identified by the observational-tests review as an open calculation would likewise be suppressed at the same parametrically small level as the particle-creation rate found here, rather than appearing at leading, potentially observable order. This is a tentative inference rather than a completed calculation, since translating a field-theoretic propagator correction into an observable photon-redshift correction requires additional steps not carried out in this paper, but it offers a provisional, quantitative expectation for the size of any such effect, where none was available before.

8. Conclusion

This paper's contributions can be summarized in four steps, each building on the last: (i) an explicit demonstration that the companion reviews' cosmological Klein-Gordon equation omits a term required by general covariance on the stated Robertson-Walker background (Section 2); (ii) a covariant completion restoring that term, verified to leave every previously established fixed-epoch result unchanged (Section 3); (iii) a reduction of the completed equation to the standard time-dependent-oscillator form used throughout the mainstream theory of quantum fields in curved spacetime, via the standard conformal-time and field-rescaling technique (Section 4); and (iv) an explicit, leading-adiabatic- order calculation of the mode functions, Bogoliubov coefficient, and two-time propagator that the companion reviews left as open questions (Sections 5–6).

This paper has taken up directly the central open question repeatedly identified across five companion reviews of the Cosmological Special Theory of Relativity: the extension of the theory's equal-time field quantization to a genuine two-time propagator. We showed first, by explicit computation, that the companion reviews' Klein-Gordon equation omits a Hubble friction term required by general covariance on the stated Robertson-Walker background, proposed a covariant completion restoring that term, and verified that this completion leaves every previously established fixed-epoch correspondence result undisturbed. We then applied the standard adiabatic mode-function and Bogoliubov formalism of the mainstream theory of quantum fields in curved spacetime to the completed equation, obtaining an explicit leading-order two-time propagator (Equation (13)) and showing that cosmological particle creation is suppressed at zeroth adiabatic order and parametrically small at the next order (Equation (12)), consistent with the extreme adiabaticity already estimated in a companion review. The precise, non-parametric evaluation of the leading Bogoliubov coefficient for a specified expansion history, and the extension of this covariant completion to the Dirac and Klein-Gordon-Maxwell sectors of the theory, are identified as the natural next steps building on the calculation presented here.

More broadly, this paper illustrates a general methodological point about extending a theory built by direct analogy with flat-space physics into a genuinely curved or dynamical background: the substitution rules that work correctly for time-independent quantities can silently fail to generate terms that only become relevant once genuine time-dependence is retained, and such omissions can remain invisible for as long as the theory is applied only within the regime – here, the fixed cosmological epoch – where the missing term vanishes identically. The gap identified in Section 2 was not detectable from within any of the extensive fixed-epoch correspondence results established in the five companion reviews; it became visible only once those reviews' own stated open question – the two-time propagator – was taken seriously enough to require solving the field equation across a finite interval of cosmological time, at which point the need for a properly covariant starting point could no longer be avoided. This suggests, as a general methodological lesson for the continued development of CSTR, that any future extension of the theory to a new physical sector should verify covariance explicitly, by the same direct computation carried out in Section 2, before proceeding to build further results on top of a substitution-rule equation whose covariance has only been assumed rather than checked.

Finally, it is worth being precise about the scope of what has been resolved. The "two-time propagator problem" as stated across the companion reviews admitted, in principle, several possible resolutions: that the fixed-epoch quantization was already secretly consistent with a trivial (identity) multi-epoch extension; that it required a substantial revision of the underlying field equation before any multi-epoch statement could be made sense of at all; or that it pointed to a genuine, observably large departure from standard physics once multi-epoch effects were included. This paper's finding sits closest to the second possibility, refined by the third at a quantitatively negligible level: the field equation did require a specific, identifiable revision (the covariant completion of Section 3), but once that revision is made, the resulting multi-epoch physics – leading-order Bogoliubov coefficients, particle creation – turns out to be parametrically tiny for ordinary matter at the present cosmological epoch, consistent with, rather than in tension with, the extensive fixed-epoch correspondence already established throughout the companion reviews. The theory is, in this sense, more completely specified after this paper than before it, without any of its previously stated physical predictions having been overturned.

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A. Glossary of Key Symbols

Symbol	Meaning
t_0, η	Cosmological time, conformal time
$\tau(t_0)$	Cosmological expansion function
$H(t_0)$	Hubble parameter, $\dot{\tau}(t_0)/\tau(t_0)$
ϕ, χ	Klein-Gordon field, conformally rescaled field
$\Box$	Covariant d'Alembertian
$\omega_k(\eta)$	Time-dependent mode frequency
$W_k(\eta)$	WKB/adiabatic frequency function
α_k, β_k	Bogoliubov coefficients
$G^{(0)}$	Leading-order two-time propagator
CSTR	Cosmological Special Theory of Relativity

Table 2: Glossary of symbols used in this paper.

B. Index of Key Equations

Table 3 indexes the principal equations of this paper by the section in which each is derived, for ease of reference.

Equation	Content
Eq. (2)	Companion reviews' (uncompleted) KG equation
Eq. (4)	Covariant d'Alembertian on Robertson-Walker
Eq. (7)	Covariantly completed KG equation (this paper)
Eq. (8)	Conformal-time mode equation
Eq. (10)	Exact WKB/adiabatic frequency equation
Eq. (11)	Bogoliubov transformation
Eq. (12)	Leading Bogoliubov coefficient estimate
Eq. (13)	Leading-order two-time propagator

Table 3: Index of principal equations derived in this paper.

References

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