Methods of inverting the Dirichlet series with applications to arithmetic functions

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Abstract

We develop a residue-based method for recovering the coefficients of a Dirichlet series from the singularities of its analytic continuation. Starting from the classical vertical-line inversion formula, we introduce a conformal transformation that converts coefficient extraction into a contour integral and yields a spectral representation in terms of poles and residues. We apply this framework to the von Mangoldt, prime characteristic, Möbius, and Euler totient functions, obtaining representations involving the zeros and poles of the Riemann zeta function. We also introduce generalized von Mangoldt functions associated with arbitrary Dirichlet series and extend the approach to finite and Dedekind zeta functions, leading to formulas for point counts, prime-ideal counting functions, and characteristic functions over number fields. Finally, we examine prime-pair, Goldbach, and prime-tuple counting functions and derive conditional error estimates under the Riemann hypothesis. These results present contour integration and zeta-function factorizations as a unified mechanism for translating analytic spectral data into arithmetic information.

Keywords: Number theory, arithmetic functions, Dirichlet series, prime numbers.

1 Introduction

Dirichlet series provide a fundamental bridge between complex analysis and arithmetic. By encoding an arithmetic function as the coefficients of a complex-valued series, they make it possible to study discrete number-theoretic information through analytic properties such as convergence, continuation, zeros, and poles. Classical examples include the Riemann zeta function, the logarithmic derivative of the zeta function

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associated with the von Mangoldt function, and quotients of zeta functions representing the Möbius and Euler totient functions. Recovering the coefficients from these analytic objects is therefore a natural inversion problem with applications throughout analytic number theory.

The usual coefficient-extraction formula averages a Dirichlet series along a vertical line in its half-plane of absolute convergence. In this paper, we reinterpret this inversion through a conformal transformation and the residue theorem. The resulting spectral representation expresses coefficients in terms of the poles of the corresponding Dirichlet series, counted within an expanding horizontal range. This viewpoint emphasizes the contribution of the singularities of the analytic continuation and provides a common framework for deriving representations of several arithmetic functions.

We apply this method first to the von Mangoldt, prime characteristic, Möbius, and Euler totient functions, obtaining formulas involving the zeros and poles of the Riemann zeta function. We then consider finite zeta functions, Euler-product expansions, and a generalized von Mangoldt function associated with an arbitrary Dirichlet series. The same perspective is extended to Dedekind zeta functions in order to describe prime-ideal counting and characteristic functions over number fields. Finally, we examine counting functions for prime pairs, Goldbach representations, and prime tuples, together with conditional error estimates under the Riemann hypothesis.

The purpose of this work is to present these constructions within a unified analytic framework. The emphasis is on showing how contour integration, residue calculations, and zeta-function factorizations can be used systematically to translate spectral information into formulas for arithmetic coefficients and counting functions. The remainder of the paper develops the inversion method, illustrates its arithmetic applications, extends it to zeta functions over other algebraic settings, and concludes with estimates for selected prime-counting problems.

2 Conformal mapping and spectral representation

This section develops a residue-theoretic representation for the coefficients of a Dirichlet series, following the approach introduced in [5]. Let $a = (a_n)_{n\geq 1}$ be an arithmetic sequence, and consider the associated Dirichlet series

$$D(a, s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad \Re(s) > \sigma_0, \tag{2.0.1}$$

in its half-plane of convergence. For any fixed $\sigma > \sigma_0$, the orthogonality of the exponential factors $e^{it\log(n/k)}$ gives the coefficient-extraction formula

$$a_n = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} n^{\sigma+it} D(a, \sigma + it)\, dt \tag{2.0.2}$$

Introduce the change of variables $z = e^{i\pi t/T}$, so that $it = (T/\pi)\log z$ on the chosen branch of the logarithm. The vertical-line average can then be written as the contour integral

$$a_n = \lim_{T\to\infty} \frac{1}{2\pi i} \oint_{|z|=1} n^{\sigma+(T/\pi)\log z} D(a, \sigma + (T/\pi)\log z)\, \frac{dz}{z} \tag{2.0.3}$$

If the integrand admits analytic continuation to a region containing a closed contour C, and the contour deformation is compatible with the chosen logarithmic branch and does not cross unaccounted singularities, the unit circle may be replaced by C:

$$a_n = \lim_{T\to\infty} \frac{1}{2\pi i} \oint_{C} n^{\sigma+(T/\pi)\log z} D(a, \sigma + (T/\pi)\log z)\, \frac{dz}{z} \tag{2.0.4}$$

Absolute convergence on the line $\Re(s) = \sigma$ provides the estimate

$$|\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} k^{\sigma+it} D(a, \sigma + it)\, dt| \le k^{\sigma} D(|a|, \sigma) \tag{2.0.5}$$

whenever $D(|a|, \sigma)$ is finite. Consequently, for fixed k, the unnormalized integral grows at most linearly with T:

$$\lim_{T\to\infty} \int_{-T}^{T} k^{\sigma+it} D(a, \sigma + it)\, dt = O(T) \tag{2.0.6}$$

Under the additional hypotheses needed to evaluate the deformed contour by residues and to control the remaining contour contributions, this leads to the spectral representation

$$a_n = \lim_{T\to\infty} \frac{\pi}{T} \sum_{\substack{|D(a,p_i)|=+\infty \\ -T\le\Im(p_i)\le T}} Res(n^s D(a, s); p_i) \tag{2.0.7}$$

where p_i ranges over the poles of $D(a, s)$ in the strip $-T \le \Im(p_i) \le T$, with multiplicities incorporated into the residues. In this representation, a nontrivial limiting contribution requires a sufficiently rich family of singularities as $T \to \infty$; in particular, the relevant imaginary parts must not remain confined to a bounded set.

2.1 Applications

We first apply the preceding framework to the von Mangoldt function. Its Dirichlet series is the logarithmic derivative of the Riemann zeta function:

$$D(\Lambda, s) = -\frac{\zeta'(s)}{\zeta(s)}. \tag{2.1.1}$$

Using the partial-fraction representation stated in [4], we write

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} - \sum_{\zeta(\rho)=0} \frac{1}{s-\rho}. \tag{2.1.2}$$

where ρ ranges over the zeros of $\zeta(s)$, including the nontrivial zeros in the critical strip and the trivial zeros $\rho = -2m$, $m \in \mathbb{N}$. Applying the residue representation in (2.0.7) gives

$$\Lambda(k) = \lim_{T\to\infty} \frac{\pi}{T}k - \frac{\pi}{T}\sum_{m=1}^{\infty} k^{-2m} - \frac{\pi}{T} \sum_{\substack{\zeta(\rho)=0 \\ 0<\Re(\rho)<1 \\ -T\le\Im(\rho)\le T}} k^{\rho} \tag{2.1.3}$$

For fixed k, the contributions from the pole at $s = 1$ and from the trivial zeros are of order T^{-1}. Hence

$$\Lambda(k) = \lim_{T\to\infty} O(\frac{1}{T}) - \frac{\pi}{T} \sum_{\substack{\zeta(\rho)=0 \\ 0<\Re(\rho)<1 \\ -T\le\Im(\rho)\le T}} k^{\rho} \tag{2.1.4}$$

Letting $T \to \infty$ removes these terms and leaves the contribution of the nontrivial zeros:

$$\Lambda(k) = \lim_{T\to\infty} -\frac{\pi}{T} \sum_{\substack{\zeta(\rho)=0 \\ 0<\Re(\rho)<1 \\ -T\le\Im(\rho)\le T}} k^{\rho} \tag{2.1.5}$$

We next consider the characteristic function of the primes,

$$\chi_{\mathbb{P}}(n) = \begin{cases} 1, & \text{if } n \in \mathbb{P}, \\ 0, & \text{otherwise,} \end{cases} \tag{2.1.6}$$

where $\mathbb{P}$ denotes the set of prime numbers. Möbius inversion of the prime-power contributions to $\Lambda(n)$ yields the representation [6]

$$\chi_{\mathbb{P}}(n) = \frac{1}{\log(n)} \sum_{\substack{m\ge1 \\ n^{1/m}\in\mathbb{N}}} \Lambda(n^{1/m})\mu(m), \tag{2.1.7}$$

where the sum is restricted to integers $m \ge 1$ for which $n^{1/m}$ is an integer. The factor $\mu(m)$ removes the contributions of higher prime powers. To incorporate this inversion into the spectral formula, define

$$F(n;s) = \sum_{m=1}^{\infty} \mu(m) n^{\frac{s}{m}} = \sum_{k=1}^{\infty} \frac{\log(n)^k}{k!\zeta(k)} s^k. \tag{2.1.8}$$

Formally interchanging the sums and expanding the exponential gives the displayed power-series expression. The estimate used below is

$$|F(n;s)| < |n^s|.$$

Combining the representation of Λ with the preceding Möbius inversion gives a formula for $\chi_{\mathbb{P}}$ in terms of the nontrivial zeros of $\zeta(s)$; see also [1]:

$$\chi_{\mathbb{P}}(n) = \lim_{T\to\infty} -\frac{\pi}{\log(n)T} \sum_{\substack{\zeta(\rho)=0 \\ -T\le\Im(\rho)\le T \\ 0<\Re(\rho)<1}} F(n;\rho), \tag{2.1.9}$$

Here the sum ranges over nontrivial zeros $\rho = \beta + i\gamma$, with $0 < \beta < 1$ and $-T \le \gamma \le T$. Within the same formalism, the expression may be extended from positive integers to positive real arguments as follows:

$$\lim_{T\to\infty} -\frac{\pi}{\log(n)T} \sum_{\substack{\zeta(\rho)=0 \\ -T\le\Im(\rho)\le T \\ 0<\Re(\rho)<1}} F(n;\rho) = \begin{cases} -1, & \text{if } n = \frac{1}{p} \ \forall p \in \mathbb{P}, \\ \infty, & \text{if } n = 1, \\ 1, & \text{if } n = p \ \forall p \in \mathbb{P}, \\ 0, & \text{otherwise ,} \end{cases} \tag{2.1.10}$$

for $n > 0$. This extension can be used to express the number $\omega(n)$ of distinct prime divisors of an integer n:

$$\omega(n) = \sum_{d=1}^{\lceil \frac{n}{2} \rceil} \chi_{\mathbb{P}}\left(\frac{n}{d}\right)$$

Equivalently, $\omega(n) = \sum_{p|n} 1$. The Möbius function is then defined by

$$\mu(n) = \begin{cases} (-1)^{\omega(n)}, & \text{if } p^2 \nmid n \, \forall p \in \mathbb{P}, \\ 1, & \text{if } n = 1, \\ 0, & \text{otherwise,} \end{cases} \tag{2.1.11}$$

where $\omega(n)$ denotes the number of distinct prime divisors of n. Its Dirichlet series is the reciprocal of the Riemann zeta function:

- For $\Re(s) > 1$,

$$\frac{1}{\zeta(s)} = D(\mu, s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \quad \Re(s) > 1. \tag{2.1.12}$$

Applying the coefficient-extraction formula to $1/\zeta(s)$ gives

$$\mu(n) = \lim_{T\to\infty} \int_{\sigma-iT}^{\sigma+iT} n^s \frac{1}{\zeta(s)} \frac{ds}{2iT} \tag{2.1.13}$$

The required absolute-convergence estimate is

$$|D(\mu, s)| \le \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^{\sigma}} \le \zeta(\sigma) \tag{2.1.14}$$

because $|\mu(n)| \leq 1$; the majorant converges for $\sigma > 1$. Near a simple zero ρ of $\zeta(s)$, the residue of $k^s/\zeta(s)$ is $k^\rho/\zeta'(\rho)$. The residue formula therefore yields

$$\mu(k) = \lim_{T\to\infty} \frac{\pi}{T} \sum_{\substack{\zeta(\rho)=0\\0<\Re(\rho)<1\\-T<\Im(\rho)<T}} \frac{k^\rho}{\zeta'(\rho)}. \tag{2.1.15}$$

The same procedure applies to Euler's totient function, which counts the positive integers not exceeding n that are relatively prime to n. It is given by

$$\varphi(n) = n\prod_{p|n}\left(1-\frac{1}{p}\right) \tag{2.1.16}$$

and its Dirichlet series is

$$D(\varphi, s) = \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}, \quad \Re(s) > 2. \tag{2.1.17}$$

For $\sigma > 2$, absolute convergence follows from

$$|D(\varphi, s)| \leq \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^\sigma} = \frac{\zeta(\sigma-1)}{\zeta(\sigma)}$$

Applying the residue calculation to $\zeta(s-1)/\zeta(s)$, in the same manner as for the Möbius function, gives the following representation in terms of the nontrivial zeros of $\zeta(s)$:

$$\varphi(k) = \lim_{T\to\infty} \frac{\pi}{T} \sum_{\substack{\zeta(\rho)=0\\0<\Re(\rho)<1\\-T<\Im(\rho)<T}} k^\rho \frac{\zeta(\rho-1)}{\zeta'(\rho)} \tag{2.1.18}$$

2.2 Examples involving finite zeta functions

Let X be a variety over the finite field with q elements, and let N_m denote the number of points of X over the degree-m extension. The associated zeta function satisfies

$$log(\zeta(X, s)) = \sum_{m=1}^{+\infty} \frac{N_m}{m} q^{-ms} \tag{2.2.1}$$

The coefficient N_m can be recovered from the logarithmic derivative by the vertical-line integral

$$N_m = \lim_{T\to+\infty} \frac{1}{log(q)} \int_{a-iT}^{a+iT} q^{ms} - \frac{\zeta'(X,s)}{\zeta(X,s)} \frac{ds}{2iT} \tag{2.2.2}$$

Using the residue formula (2.0.7), this becomes

$$N_m = \lim_{T\to\infty} \frac{\pi}{T} \frac{1}{log(q)} \sum_{\substack{|\zeta(X,p_i)|=+\infty\\-T\leq\Im(p_i)\leq T}} q^{mp_i} - \frac{\pi}{T} \frac{1}{log(q)} \sum_{\substack{\zeta(X,z_i)=0\\-T\leq\Im(z_i)\leq T}} q^{mz_i} \tag{2.2.3}$$

where the poles p_i and zeros z_i are written with their multiplicities; the displayed formula corresponds to the case of simple zeros and poles. A rational factorization of the finite zeta function may be written in the form

$$log(\zeta(X,s)) = \sum_{i=0}^{2n} (-1)^{i+1} log(P_i(q^{-s})) \tag{2.2.4}$$

with factors satisfying

$$log(P_i(q^{-s})) = \sum_j log(1 - \frac{a_{ij}}{q^s})$$

Applying the same coefficient comparison to each factor $P_i(q^{-s})$ relates the power sums of its reciprocal roots a_{ij} to the zeros and poles of the corresponding function:

$$-\sum_j (a_{ij})^m = \lim_{T\to\infty} \frac{\pi}{T} \frac{1}{log(q)} \sum_{\substack{|P_i(q^{-p_k})|=+\infty \\ -T\le\Im(p_k)\le T}} q^{mp_k} - \frac{\pi}{T} \frac{1}{log(q)} \sum_{\substack{P_i(q^{-z_k})=0 \\ -T\le\Im(z_k)\le T}} q^{mz_k} \tag{2.2.5}$$

2.3 Representations from Euler products

Euler products encode arithmetic information locally, one prime at a time. The frequencies $2\pi k/\log p$ associated with a prime p give rise to the oscillatory factors

$$n^{2\pi ik/\log p} = \exp\left(2\pi ik \frac{\log n}{\log p}\right).$$

Averaging these factors over symmetric ranges of k acts as an orthogonality relation and isolates the corresponding divisibility and prime-power conditions. This observation leads to formal spectral representations for several familiar arithmetic functions. Their use below presupposes that the relevant Dirichlet series have been analytically continued to the points at which they are evaluated and that the displayed limits and sums may be interchanged.

We begin with the divisor function $d(n) = \sum_{d|n} 1$. In terms of the Euler-product spectrum, it is represented by

$$d(n) = \lim_{N\to\infty} \frac{1}{2N} \sum_{-N\le k\le N} \sum_{p\in\mathbb{P}} \frac{\log n}{\log p} n^{2\pi ik/\log p} D\left(\mathbb{1}_{(p,\cdot)=1}, \frac{2\pi ik}{\log p}\right)^2, \tag{2.3.1}$$

where

$$D\left(\mathbb{1}_{(p,\cdot)=1}, s\right) = \sum_{\substack{r\ge 1 \\ (p,r)=1}} \frac{1}{r^s}.$$

The same orthogonality principle gives the following representation of the integer characteristic function [5]:

$$\mathbb{1}_{\mathbb{Z}}(n) = \lim_{N\to\infty} \frac{1}{2N} \sum_{-N\le k\le N} \sum_{p\in\mathbb{P}} n^{2\pi ik/\log p} \prod_{\substack{q\in\mathbb{P}\\ q\neq p}} D\left(\mathbb{1}_{(p,\cdot)=1}, \frac{2\pi ik}{\log p}\right). \tag{2.3.2}$$

For the von Mangoldt function, the local prime-power contributions take the particularly simple form

$$\Lambda(n) = \lim_{N\to\infty} \frac{1}{2N} \sum_{-N\le k\le N} \sum_{p\in\mathbb{P}} \log p\, n^{2\pi ik/\log p}. \tag{2.3.3}$$

Combining this expression with the Möbius inversion encoded by F, as defined in (2.1.8), yields the corresponding representation of the prime characteristic function:

$$\chi_{\mathbb{P}}(n) = \lim_{N\to\infty} \frac{1}{2N} \sum_{-N\le k\le N} \sum_{p\in\mathbb{P}} F\left(n; \frac{2\pi ik}{\log p}\right). \tag{2.3.4}$$

The construction also applies to indicator functions for divisibility. For a fixed positive integer m, the divisor indicator introduced in [6] satisfies the formal expansion

$$\mathbb{1}_{m|n} = \lim_{N\to\infty} \frac{1}{2N} \sum_{-N\le k\le N} \sum_{p\in\mathbb{P}} \left(\frac{n}{m}\right)^{2\pi ik/\log p} D\left(\mathbb{1}_{(p,\cdot)=1}, \frac{2\pi ik}{\log p}\right). \tag{2.3.5}$$

Finally, the same method gives a representation for the greatest-common-divisor indicator:

$$\mathbb{1}_{(n,\ell)=m} = \lim_{N\to\infty} \frac{1}{2N} \sum_{-N\le k\le N} \sum_{p\in\mathbb{P}} \left(\frac{n}{m}\right)^{2\pi ik/\log p} D\left(\mathbb{1}_{(p,\cdot)=m}, \frac{2\pi ik}{\log p}\right), \tag{2.3.6}$$

where

$$D\left(\mathbb{1}_{(p,\cdot)=m}, s\right) = \sum_{\substack{r\ge 1\\ (p,r)=m}} \frac{1}{r^s}.$$

Thus, the Euler-product viewpoint converts local conditions at primes into discrete frequency averages. The formulas in this subsection are understood through the analytic continuations of $D(\mathbb{1}_{(p,\cdot)=1}, s)$ and $D(\mathbb{1}_{(p,\cdot)=m}, s)$, together with the convergence assumptions stated above.

3 Special von Mangoldt functions

The classical von Mangoldt function arises from the logarithmic derivative of the Riemann zeta function. This construction extends naturally to a Dirichlet series $D(a,s)$. Assume that $D(a,s)$ is meromorphic in the region under consideration, is nonzero on the initial line of integration, and is normalized so that its logarithmic derivative admits a Dirichlet-series expansion. We define the generalized von Mangoldt function $\Lambda_a(n)$ by extracting the coefficients of $-D'(a,s)/D(a,s)$:

$$\Lambda_a(n) = \lim_{T\to+\infty} \frac{1}{2iT} \int_{\sigma-iT}^{\sigma+iT} n^s \left(-\frac{D'(a,s)}{D(a,s)}\right) ds \tag{3.0.1}$$

The logarithmic derivative has simple poles at the zeros and poles of $D(a,s)$, even when those zeros and poles have higher multiplicity. Applying the residue representation therefore gives

$$\Lambda_a(n) = \lim_{T\to\infty} \frac{\pi}{T} \sum_{\substack{|D(a,p_i)|=+\infty \\ -T\le\Im(p_i)\le T}} b_i n^{p_i} - \frac{\pi}{T} \sum_{\substack{D(a,z_i)=0 \\ -T\le\Im(z_i)\le T}} m_i n^{z_i} \tag{3.0.2}$$

where p_i and z_i denote the poles and zeros of $D(a,s)$, while b_i and m_i are their respective positive multiplicities. Equivalently,

$$b_i = Res\left(-\frac{D'(a,s)}{D(a,s)}; p_i\right)$$

and

$$m_i = -Res\left(-\frac{D'(a,s)}{D(a,s)}; z_i\right).$$

Thus, poles of $D(a,s)$ contribute with a positive sign, whereas zeros contribute with a negative sign.

Several standard arithmetic functions illustrate this definition. For the k-fold divisor function $d_k(n)$, whose Dirichlet series is $\zeta(s)^k$, one obtains

$$\Lambda_{d_k}(n) = k\Lambda(n)$$

where $\Lambda(n)$ is the classical von Mangoldt function and

$$d_k(n) = \sum_{a_1 a_2 \cdots a_k = n} 1$$

counts the ordered factorizations of n into k positive integer factors. For the generalized sum-of-divisors function

$$\sigma_a(n) = \sum_{d|n} d^a,$$

the associated generalized von Mangoldt function is

$$\Lambda_{\sigma_a}(n) = (n^a + 1)\Lambda(n)$$

Finally, the Dirichlet series of Euler's totient function gives

$$\Lambda_\phi(n) = (n-1)\Lambda(n)$$

where $\varphi(n)$ denotes Euler's totient function.

Conversely, the coefficients a_n may be reconstructed from $\Lambda_a(n)$. Expanding the exponential of the logarithmic Dirichlet series produces

$$a_n = \frac{\Lambda_a(n)}{\log(n)} + \frac{1}{2!}\sum_{bc=n}\frac{\Lambda_a(b)\Lambda_a(c)}{\log(b)\log(c)} + \frac{1}{3!}\sum_{bcd=n}\frac{\Lambda_a(b)\Lambda_a(c)\Lambda_a(d)}{\log(b)\log(c)\log(d)} + \cdots, \quad n \geq 2. \tag{3.0.3}$$

Equivalently, the original Dirichlet series is recovered through the exponential identity

$$D(a, s) = \exp\left(\sum_{n\geq 2}\frac{\Lambda_a(n)}{\log(n)}\frac{1}{n^s}\right).$$

This identity shows that $\Lambda_a(n)$ records the prime-power-type data encoded by the logarithm of $D(a, s)$, while the preceding convolution expansion reconstructs the full coefficient sequence.

4 Prime-counting functions over number fields

Let K be a number field with ring of integers O_K. The Dedekind zeta function of K is defined, for $\Re(s) > 1$, by

$$\zeta_K(s) = \sum_{\mathfrak{a}\neq(0)}\frac{1}{N(\mathfrak{a})^s}, \tag{4.0.1}$$

where the sum extends over the nonzero integral ideals $\mathfrak{a} \subseteq O_K$, and $N(\mathfrak{a}) = |O_K/\mathfrak{a}|$ denotes the absolute ideal norm. Unique factorization of ideals gives the Euler product

$$\zeta_K(s) = \prod_{\mathfrak{p}}\left(1 - \frac{1}{N(\mathfrak{p})^s}\right)^{-1}, \tag{4.0.2}$$

where $\mathfrak{p}$ ranges over the nonzero prime ideals of O_K.

The corresponding prime-ideal counting function is

$$\pi_K(x) = \sum_{N(\mathfrak{p})\leq x} 1. \tag{4.0.3}$$

It is useful to introduce the weighted prime-power counting function

$$J_K(x) = \sum_{m\geq 1}\frac{1}{m}\sum_{N(\mathfrak{p})^m\leq x} 1. \tag{4.0.4}$$

Möbius inversion then yields

$$\pi_K(x) = \sum_{m\geq 1}\frac{\mu(m)}{m}J_K(x^{1/m}). \tag{4.0.5}$$

This is the number-field analogue of the classical relation between the functions $\pi(x)$ and $J(x)$.

The Euler product also gives

$$\log \zeta_K(s) = \sum_{\mathfrak{p}} \sum_{m\geq 1} \frac{1}{mN(\mathfrak{p})^{ms}}, \qquad \Re(s) > 1. \tag{4.0.6}$$

Consequently, Perron's formula represents $J_K(x)$, at points of continuity and for $c > 1$, as

$$J_K(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^s}{s} \log \zeta_K(s)\, ds. \tag{4.0.7}$$

To relate this integral to the spectral data of ζ_K, suppose that an appropriate canonical product has been chosen and write it schematically as

$$\zeta_K(s) = e^{Z(s)} \prod_{\zeta_K(z_i)=0} \left(1 - \frac{s}{z_i}\right)^{d_i'} \prod_{|\zeta_K(p_i)|=\infty} \left(1 - \frac{s}{p_i}\right)^{-d_i}, \tag{4.0.8}$$

where $Z(s)$ contains the entire factors, while z_i and p_i denote the zeros and poles with multiplicities d_i' and d_i, respectively. After fixing consistent branches and accounting for the entire and gamma factors, the identity

$$-\operatorname{Li}(x^b) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^s}{s} \log\left(1 - \frac{s}{b}\right) ds \tag{4.0.9}$$

leads formally to an explicit formula of the form

$$J_K(x) = A + \sum_{|\zeta_K(p_i)|=\infty} d_i \operatorname{Li}(x^{p_i}) - \sum_{\zeta_K(z_i)=0} d_i' \operatorname{Li}(x^{z_i}), \tag{4.0.10}$$

where A, together with any additional elementary terms, depends on the normalization and on the factors absorbed into $Z(s)$. For complex exponents, $\operatorname{Li}(x^s)$ is understood by analytic continuation with a fixed branch.

Finally, define the characteristic function of the set of prime-ideal norms by

$$\chi_K(n) = \begin{cases} 1, & \text{if } n = N(\mathfrak{p}) \text{ for some prime ideal } \mathfrak{p}, \\ 0, & \text{otherwise.} \end{cases} \tag{4.0.11}$$

Within the residue formalism developed above, its spectral representation is formally

$$\begin{aligned} \chi_K(x) = {} & \lim_{T\to\infty} \frac{\pi}{T\log x} \sum_{\substack{|\zeta_K(p_i)|=\infty \\ -T\leq \Im(p_i)\leq T}} d_i F(x; p_i) \\ & - \lim_{T\to\infty} \frac{\pi}{T\log x} \sum_{\substack{\zeta_K(z_i)=0 \\ -T\leq \Im(z_i)\leq T}} d_i' F(x; z_i), \end{aligned} \tag{4.0.12}$$

where $F(x; s)$ is defined in (2.1.8). As in the preceding sections, this expression requires convergence of the truncated sums and control of the remaining contour contributions.

5 Counting selected prime configurations

This section introduces counting functions for prime pairs, Goldbach representations, and prime k-tuples. In each case, the exact discrete count can be written as a Stieltjes integral with respect to the prime-counting function. The logarithmic-integral expressions displayed below should be understood as conditional models: the Riemann hypothesis alone does not imply asymptotic formulas for fixed prime gaps, Goldbach representations, or general prime tuples. The stated error terms therefore depend on the additional pointwise and Stieltjes assumptions discussed in Section 7.

5.1 Prime-pair counting function

Fix an integer $g \geq 1$. We wish to count primes $p \leq x$ for which $p + g$ is also prime. The corresponding counting function is

$$\pi_g(x) = \sum_{n \leq x} \chi_{\mathbb{P}}(n)\chi_{\mathbb{P}}(n+g). \tag{5.1.1}$$

Because integration with respect to $d\pi(t)$ restricts the variable t to prime values, the same count has the exact Stieltjes representation

$$\pi_g(x) = \int_2^x \chi_{\mathbb{P}}(t+g)\, d\pi(t). \tag{5.1.2}$$

Replacing each prime indicator by its logarithmic density and invoking the additional assumptions described in Section 7 leads to the conditional approximation

$$\pi_g(x) = \int_2^x \frac{dt}{\log t\, \log(t+g)} + O\Big(\sqrt{x+g}\log(x+g)\Big), \tag{5.1.3}$$

Every prime other than 2 is odd. Hence, if g is odd, the only possible pair is $(2, 2+g)$. With the convention that $u(y) = 1$ for $y \geq 0$ and $u(y) = 0$ otherwise, the count reduces to

$$\pi_g(x) = \chi_{\mathbb{P}}(2+g)u(x-2), \qquad g \in 2\mathbb{N}+1. \tag{5.1.4}$$

Thus odd gaps are governed entirely by whether $2 + g$ is prime, whereas even gaps may occur between two odd primes.

5.2 Goldbach representation function

For an even integer $2m$, define $G_2(m)$ to be the number of ordered prime representations $2m = p + q$. Explicitly,

$$G_2(m) = \sum_{2 \leq n \leq 2m-2} \chi_{\mathbb{P}}(n)\chi_{\mathbb{P}}(2m-n). \tag{5.2.1}$$

The equivalent Stieltjes representation is

$$G_2(m) = \int_2^{2m-2} \chi_{\mathbb{P}}(2m-t)\, d\pi(t). \tag{5.2.2}$$

Under the same additional hypotheses used for the prime-pair model, one obtains the conditional estimate

$$G_2(m) = \int_2^{2m-2} \frac{dt}{\log t\, \log(2m-t)} + O\Big(\sqrt{2m-2}\log(2m-2)\Big). \tag{5.2.3}$$

This function counts ordered pairs; an unordered representation count may be obtained by accounting for the symmetry $p \leftrightarrow q$, with separate treatment of the diagonal case $p = q = m$.

5.3 Prime k-tuple counting function

Let $0 = g_0 < g_1 < \cdots < g_{k-1}$ be fixed integer offsets. A prime k-tuple with this pattern is a collection $p, p+g_1, \ldots, p+g_{k-1}$ in which every entry is prime. Equivalently, if $\vec{p} = (p_1, \ldots, p_k)$, then

$$p_j - p_1 = g_{j-1}, \qquad j = 1, 2, \ldots, k. \tag{5.3.1}$$

The associated counting function is

$$\pi_{(g_1, g_2, \ldots, g_{k-1})}(x) = \sum_{n \le x} \chi_{\mathbb{P}}(n) \prod_{i=1}^{k-1} \chi_{\mathbb{P}}(n+g_i). \tag{5.3.2}$$

As before, this can be written exactly as a Stieltjes integral:

$$\pi_{(g_1, g_2, \ldots, g_{k-1})}(x) = \int_2^x \prod_{i=1}^{k-1} \chi_{\mathbb{P}}(t+g_i)\, d\pi(t). \tag{5.3.3}$$

The corresponding logarithmic-density model, subject to the stronger assumptions stated above, is

$$\pi_{(g_1, g_2, \ldots, g_{k-1})}(x) = \int_2^x \frac{1}{\log t} \prod_{i=1}^{k-1} \frac{1}{\log(t+g_i)}\, dt + O\Big(\sqrt{x+g}\log(x+g)\Big). \tag{5.3.4}$$

where $g = \max\{g_i : 1 \le i \le k-1\}$. For a nontrivial asymptotic count, the offset set must also satisfy the usual local admissibility conditions; otherwise a congruence obstruction may force the counting function to remain bounded or vanish.

6 Estimates for the prime-gap function

Let p_n denote the n-th prime and let $g_n = p_{n+1} - p_n$ be the corresponding prime gap. We study the Dirichlet generating function

$$D(g, s) = \sum_{n=1}^{\infty} \frac{g_n}{n^s} = \sum_{n=2}^{\infty} \frac{1}{\pi(n)^s}, \tag{6.0.1}$$

where the second expression follows by grouping integers according to the constant values of the step function $\pi(n)$. The prime number theorem gives $\pi(x) \sim x/\log x$ [4]; substituting this average-order approximation suggests

$$D(g,s) \sim \sum_{n=1}^{\infty} \frac{(\log n)^s}{n^s}, \qquad \Re(s) > 1. \tag{6.0.2}$$

The step-function representation also has the integral form

$$D(g,s) = \int_2^{\infty} \frac{1}{(\pi(x))^s}\, dx, \tag{6.0.3}$$

Replacing $\pi(x)$ by its prime-number-theorem approximation then gives

$$D(g,s) \sim \int_1^{\infty} \frac{(\log x)^s}{x^s}\, dx. \tag{6.0.4}$$

After the substitution $u = (s-1)\log x$, the integral evaluates to

$$D(g,s) \sim \frac{\Gamma(s+1)}{(s-1)^{s+1}}. \tag{6.0.5}$$

Under the hypotheses required for Mellin inversion, the inverse prime-counting function can be represented by

$$\pi^{-1}(x) = 2 + \int_{2-i\infty}^{2+i\infty} \frac{(x-1)^s}{s} D(g,s)\, \frac{ds}{2\pi i}, \qquad x > 1. \tag{6.0.6}$$

Substituting the asymptotic expression for $D(g,s)$ and expanding the inverse Mellin integral yields

$$\pi^{-1}(x) \sim 2 + (x-1)\log(x-1) - (x-1)\sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} \frac{1}{(x-1)^k}, \qquad x > 1 + e. \tag{6.0.7}$$

The same contour integral can be written in closed form using the lower real branch W_{-1} of the Lambert W function [3]:

$$\int_{2-i\infty}^{2+i\infty} \frac{(x-1)^s}{s} \frac{\Gamma(s+1)}{(s-1)^{s+1}} \frac{ds}{2\pi i} = -(x-1)W_{-1}\left(\frac{1}{1-x}\right), \tag{6.0.8}$$

Consequently,

$$\pi^{-1}(x) \sim 2 - (x-1)W_{-1}\left(\frac{1}{1-x}\right), \qquad x > 1 + e. \tag{6.0.9}$$

The average-order relation between consecutive primes and the inverse counting function may be expressed as [6]

$$g_n \sim \frac{d}{dx}\pi^{-1}(x+1)\Big|_{x=n}.$$

Differentiating the series representation gives

$$g_n \sim \log(n) + 1 - \sum_{k=1}^{\infty} \frac{k^{k-1}(1-k)}{k!} \frac{1}{n^k}, \qquad n > e. \tag{6.0.10}$$

whereas differentiating the Lambert-W form gives the equivalent approximation

$$g_n \sim -\frac{(W_{-1}(-\frac{1}{n}))^2}{1 + W_{-1}(-\frac{1}{n})}, \qquad n > e. \tag{6.0.11}$$

To obtain a conditional refinement, assume the Riemann hypothesis and use a bound of the form

$$\pi(x) \le \frac{x}{\log x} + c\sqrt{x}\log x$$

for some constant $c > 0$. For positive real s, this upper bound for $\pi(x)$ gives the lower estimate

$$|D(g,s)| \ge |\int_1^{+\infty} (\frac{x}{log(x)} + c\sqrt{x}log(x))^{-s}\, dx| \tag{6.0.12}$$

Expanding the integrand and evaluating the resulting terms leads formally to

$$|D(g,s)| \ge |\sum_{k=0}^{+\infty} c^k \frac{\Gamma(1-s)}{k!\Gamma(1-s-k)} \frac{\Gamma(s+2k+1)}{(s-1+\frac{k}{2})^{s+2k+1}}| \tag{6.0.13}$$

7 Conditional error estimates under the Riemann hypothesis

Throughout this section, we assume the Riemann hypothesis. We first derive a transform estimate from the classical bound for the Chebyshev function $\psi(x)$. It is important to distinguish this summatory estimate from a pointwise estimate: the Riemann hypothesis controls $\psi(x) - x$, but it does not by itself imply a pointwise asymptotic formula for $\Lambda(n)$ or $\chi_{\mathbb{P}}(n)$. Accordingly, the later pointwise and prime-pattern estimates require the additional coefficient-extraction assumptions stated below.

For $\Re(s) > 0$, Stieltjes integration gives

$$\sum_{n\ge 2} (\Lambda(n) - 1)e^{-ns} = \int_0^{+\infty} e^{-ts}\, d(\psi(t) - \lfloor t \rfloor) \tag{7.0.1}$$

Integration by parts, with the boundary terms vanishing in the region of convergence, yields

$$\sum_{n\ge 2} (\Lambda(n) - 1)e^{-ns} = s\int_0^{+\infty} e^{-ts}(\psi(t) - \lfloor t \rfloor)\, dt \tag{7.0.2}$$

Under the Riemann hypothesis, the estimate $\psi(t) - \lfloor t \rfloor = O(\sqrt{t}\log^2 t)$ holds in the required asymptotic sense; see [2]. Therefore,

$$\left| s\int_0^{+\infty} e^{-ts}(\psi(t) - \lfloor t \rfloor)\, dt \right| \le \left| cs\int_0^{+\infty} e^{-ts}\sqrt{t}\log^2(t)\, dt \right|. \tag{7.0.3}$$

Differentiating the standard gamma-integral identity with respect to its parameter evaluates the majorant as

$$s\int_0^{+\infty} e^{-ts}\sqrt{t}\log^2(t)\, dt = \left(\Gamma''(\frac{3}{2}) - 2\Gamma'(\frac{3}{2})\log(s) + \Gamma(\frac{3}{2})\log^2(s)\right)\frac{1}{\sqrt{s}}. \tag{7.0.4}$$

where $\Gamma'(x)$ and $\Gamma''(x)$ denote the first and second derivatives of the gamma function. For convenience, define

$$f(s) = \left(\Gamma''(\frac{3}{2}) - 2\Gamma'(\frac{3}{2})\log(s) + \Gamma(\frac{3}{2})\log^2(s)\right)\frac{1}{\sqrt{s}}.$$

To pass from this transform bound to an estimate for individual coefficients, consider the following contour-extraction expression:

$$\Lambda(n) - 1 \le \lim_{\epsilon\to 0}\frac{c}{2\pi i}\oint_{1+\epsilon e^{it}} z^{n-1}f(\log z)\, dz \tag{7.0.5}$$

Using $\log z \sim z - 1$ as $z \to 1$, the proposed local residue calculation becomes

$$\lim_{\epsilon\to 0}\frac{c}{2\pi i}\oint_{1+\epsilon e^{it}} z^{n-1}f(\log z)\, dz = c\ \mathrm{Res}(e^{sn}f(s); 0). \tag{7.0.6}$$

Formally evaluating this residue gives

$$c\ \mathrm{Res}(e^{sn}f(s); 0) = c\frac{\log^2(n)}{\sqrt{n}}. \tag{7.0.7}$$

This coefficientwise conclusion does not follow from the Riemann hypothesis alone. If the contour extraction and the required uniform estimates are additionally assumed to be valid, then one obtains the pointwise bound

$$\Lambda(n) - 1 \le c\frac{\log^2(n)}{\sqrt{n}}. \tag{7.0.8}$$

for some constants $c > 0$ and N_0, and all $n > N_0$. Under this additional pointwise hypothesis, the inequality $\chi_{\mathbb{P}}(n) \le \Lambda(n)/\log n$ gives

$$\chi_{\mathbb{P}}(n) - \frac{1}{\log n} \le c\frac{\log n}{\sqrt{n}}. \tag{7.0.9}$$

After enlarging the constant if necessary, it is convenient to use the slightly weaker estimate

$$\chi_{\mathbb{P}}(n) - \frac{1}{\log n} \le c\frac{\log n + 2}{\sqrt{n}}. \tag{7.0.10}$$

We now propagate this assumed pointwise estimate to several prime-pattern counting functions. For a fixed shift g, recall the Stieltjes representation

$$\pi_g(x) = \int_2^x \chi_{\mathbb{P}}(t+g)\, d\pi(t) \tag{7.0.11}$$

Subtracting the logarithmic-integral model gives

$$\left|\pi_g(x) - \int_2^x \frac{dt}{\log(t+g)\log t}\right| \le \left|\int_2^x \chi_{\mathbb{P}}(t+g)\, d\pi(t) - \int_2^x \frac{dt}{\log(t+g)\log t}\right|. \tag{7.0.12}$$

The triangle inequality separates the error into a pointwise prime-indicator term and a prime-counting term:

$$\left|\pi_g(x) - \int_2^x \frac{dt}{\log(t+g)\log t}\right| \le \left|\int_2^x \left(\chi_{\mathbb{P}}(t+g) - \frac{1}{\log(t+g)}\right) d\pi(t)\right| + E(x;g)$$

where

$$E(x;g) = \left|\int_2^x \frac{1}{\log(t+g)}\, d\left(\pi(t) - \mathrm{Li}(t)\right)\right|.$$

The assumed pointwise estimate for $\chi_{\mathbb{P}}$ implies

$$\left|\int_2^x \left(\chi_{\mathbb{P}}(t+g) - \frac{1}{\log(t+g)}\right) d\pi(t)\right| \le \left|c\int_2^x \frac{\log(t+g)}{\sqrt{t+g}}\, d\lfloor t\rfloor\right|. \tag{7.0.13}$$

Comparison with the corresponding continuous integral gives

$$\left|c\int_2^x \frac{\log(t+g)}{\sqrt{t+g}}\, d\lfloor t\rfloor\right| \le c\sqrt{x+g}\log(x+g). \tag{7.0.14}$$

Moreover, $1/\log(t+g) \le 1$ for $t+g \ge 3$. Using the Riemann-hypothesis estimate for the prime-counting error, interpreted with the required Stieltjes control, gives

$$\left|\int_2^x \frac{1}{\log(t+g)}\, d\left(\pi(t) - \mathrm{Li}(t)\right)\right| \le \left|\pi(x) - \mathrm{Li}(x)\right|. \tag{7.0.15}$$

By [2], this is bounded conditionally by

$$|\pi(x) - \mathrm{Li}(x)| \le c'\sqrt{x}\log x \le c'\sqrt{x+g}\log(x+g).$$

Combining the two contributions yields

$$\left|\pi_g(x) - \int_2^x \frac{dt}{\log(t+g)\log t}\right| \le c_3\sqrt{x+g}\log(x+g). \tag{7.0.16}$$

An analogous decomposition applies to the Goldbach prime-pair counting function:

$$\left|G_2(m) - \int_2^{2m-2} \frac{1}{\log(2m-t)\log t}\, dt\right| \tag{7.0.17}$$

$$= \left|\int_2^{2m-2} \chi_{\mathbb{P}}(2m-t)\, d\pi(t) - \int_2^{2m-2} \frac{1}{\log(2m-t)\log t}\, dt\right|. \tag{7.0.18}$$

Applying the triangle inequality gives

$$\left|\int_2^{2m-2} \chi_{\mathbb{P}}(2m-t)\, d\pi(t) - \int_2^{2m-2} \frac{1}{\log(2m-t)\log t}\, dt\right|$$
$$\leq \left|\int_2^{2m-2} \chi_{\mathbb{P}}(2m-t) - \frac{1}{\log(2m-t)}\, d\pi(t)\right| + |R(m)|.$$

Here

$$R(m) = \int_2^{2m-2} \frac{1}{\log(2m-t)}\, d\left(\pi(t) - \mathrm{Li}(t)\right).$$

Since $1/\log(2m-t) \leq 1/\log 2$ on the interval of integration, the assumed Stieltjes estimate gives

$$|R(m)| \leq c\sqrt{2m-2}\log(2m-2).$$

The pointwise prime-indicator contribution is bounded in the same way:

$$\int_2^{2m-2} \chi_{\mathbb{P}}(2m-t) - \frac{1}{\log(2m-t)}\, d\pi(t) \leq c\sqrt{2m-2}\log(2m-2)$$

Consequently,

$$|G_2(m) - \int_2^{2m-2} \frac{1}{\log(2m-t)\log t}\, dt| \leq c'\sqrt{2m-2}\log(2m-2). \tag{7.0.19}$$

Repeating the decomposition for each shift gives the corresponding conditional estimate for a prime k-tuple:

$$\left|\pi_{(g_1,g_2,\ldots,g_{k-1})}(x) - \int_2^x \frac{1}{\log t}\prod_{i=1}^{k-1}\frac{1}{\log(t+g_i)}\, dt\right| = E(x). \tag{7.0.20}$$

Since $1/\log(t+g_i) < 1$ whenever $t + g_i \geq 3$, the separate contributions satisfy

$$E(x) \leq \sum_{i=0}^{k-1} c_i\sqrt{x+g_i}\log(x+g_i)$$

where $g_0 = 0$. Writing $g = \max\{g_i : 1 \leq i \leq k-1\}$, these terms may be combined into

$$E(x) \leq c''\sqrt{x+g}\log(x+g).$$

Thus, the final prime-pair and prime-tuple estimates are conditional not only on the Riemann hypothesis, but also on the additional pointwise and Stieltjes assumptions used to pass from summatory information to individual prime-pattern counts.

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