A Geometric Explanation of Dark Matter Based on General Relativity

Hao Shen * Ruipeng Ma

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Abstract

Within the framework of standard general relativity[1], under the assumptions of staticity, spherical symmetry and the strong energy condition, we prove that a geometric transition zone — the "reverse-bending zone" — must appear in the periphery of any finite self-gravitating system, where the t–r sectional curvature changes sign from negative to positive. This zone is bounded by the curvature zero r_0, the curvature peak r_{peak}, and the matter boundary R; in the interval (r_0, R) the sectional curvature smoothly transforms from matter-dominated spherical compression to vacuum saddle-shaped stretching. The reverse-bending zone is not a free vacuum but a forced geometry locked jointly by the interior baryonic potential well and the far-field boundary condition. Within this zone, the Misner–Sharp-type gravitational mass $M(r)$ continues to grow: it grows faster than linearly in the region $r_0 \to r_{\text{peak}}$, and although the growth slows down in the region $r_{\text{peak}} \to R$, it never ceases. The resulting geometric Weyl stretching together with the self-energy of the gravitational field provide an extra centripetal acceleration, which naturally manifests itself, in the weak-field approximation, as an approximately logarithmic potential and a flattening of the rotation curves. The theory yields parameter-free, falsifiable predictions that can be directly tested with existing rotation-curve and photometric data. These results show that, without introducing new particles or modifying the field equations, the forced geometry within general relativity can produce "dark-matter-like" gravitational effects on galactic scales.

Keywords: general relativity, dark matter, sectional curvature, continuous metric, reverse-bending zone

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*Corresponding author, email: shenhao0119@163.com

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1 Introduction

The nature of dark matter is one of the deepest unsolved mysteries in fundamental physics. The standard ΛCDM paradigm attributes the phenomenon to unknown particle species. However, after decades of intense experimental searches, dark-matter particles have not been detected directly. Moreover, all the various theoretical explanations for dark matter have so far not been accepted by the scientific community as a perfect description.

In this paper we find an alternative and understandable path to the problem from within relativity. We prove that, without introducing any new physical entity or free parameter, the mathematical structure of general relativity already contains a geometric mechanism capable of producing the observational features of dark matter[6]. The key point is that we use standard relativistic methods to demonstrate that, when the spacetime curvature around a matter aggregation transitions to the far field, a saddle-shaped geometric transition zone necessarily forms in the spacetime outside the matter. This saddle-shaped spacetime geometry is precisely the geometric origin of the “dark-matter” effect.For standard references on general relativity, see [2, 3, 5].

2 Geometric Transition Theorem

2.1 Prerequisites

We consider a static, spherically symmetric, self-gravitating bound system described by Einstein’s general relativity, satisfying the following physically realizable and mathematically self-consistent conditions:

1. **Finite, smooth distribution of matter.** The energy-momentum tensor $T_{\mu\nu}$ is confined within a sphere of coordinate radius $r = R$. At the boundary $r = R$, the energy density $\rho(r)$ and radial pressure $p(r)$ satisfy

$$\lim_{r\to R^-} \rho(r) = 0, \qquad \lim_{r\to R^-} p(r) = 0,$$

and $\rho(r), p(r) \in C^1[0, R]$ (first-order derivatives continuous). The system possesses a finite positive ADM mass $M > 0$, consistent with the integrated interior mass:

$$M = \lim_{r\to\infty} \frac{c^2 r}{2G}\left(1 - e^{-2\lambda(r)}\right) = \int_0^\infty 4\pi r^2 \rho(r)\, dr.$$

The equation of state $p = p(\rho)$ satisfies the causality condition $0 \le dp/d\rho \le c^2$. This theorem applies to self-gravitating bound systems composed of normal matter (stars, galaxies, galaxy clusters), and excludes systems that contain spacetime singularities, such as black holes or naked singularities. For systems like galaxies, whose evolutionary timescale is much longer than the dynamical timescale, the static and spherical symmetry assumption is a good approximation.

2. **Strong energy condition.** For all timelike vector fields u^μ, the energy-momentum tensor satisfies

$$T_{\mu\nu} u^\mu u^\nu \ge \frac{1}{2} T^\lambda_{\ \lambda} u^\sigma u_\sigma.$$

For a static spherically symmetric ideal fluid (the most general form of matter compatible with spherical symmetry), this condition is equivalent to

$$\rho c^2 + 3p \geq 0.$$

The strong energy condition guarantees the attractive nature of gravity. All known macroscopic normal matter (stars, gas, plasma) strictly satisfies this condition.

3. **Regularity of spacetime.** The spacetime metric tensor $g_{\mu\nu} \in C^2(\mathcal{M})$, i.e., the metric and its first and second derivatives are continuous throughout the entire manifold $\mathcal{M}$. This ensures that the Riemann curvature tensor $R_{\mu\nu\rho\sigma}$ is globally well-defined and all its components are bounded, ruling out the unphysical situation of a singular thin shell of curvature (which would correspond to infinite stress) at the boundary $r = R$.

4. **Exterior vacuum.** In the region $r > R$, $T_{\mu\nu} = 0$, and the spacetime satisfies the vacuum Einstein field equations with a cosmological constant $\Lambda \geq 0$:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0,$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ is the Einstein tensor and Λ has the dimension of m^{-2}.

5. **Asymptotic behaviour.** As $r \to \infty$, the spacetime tends asymptotically to Minkowski spacetime (if $\Lambda = 0$) or to de Sitter spacetime (if $\Lambda > 0$).

2.2 Statement of the Theorem

Theorem 2.1 (Geometric Transition Theorem)**.** *Under the above premises, there necessarily exists a* ***unique critical radius*** r_0 *inside the sphere, satisfying* $0 < r_0 < R$*, such that the sectional curvature of the* t*-*r *plane in an orthonormal frame,*

$$K_{tr}(r) \equiv R_{\hat{t}\hat{r}\hat{t}\hat{r}}(r),$$

vanishes at that point, i.e., $K_{tr}(r_0) = 0$*. Moreover:*

- *In the region* $0 < r < r_0$*,* $K_{tr}(r) < 0$*, corresponding to* ***spherical curvature*** *(radial compression, characteristic of a matter-dominated gravitational potential well);*
- *In the region* $r_0 < r < R$*,* $K_{tr}(r) > 0$*, corresponding to* ***saddle-shaped curvature*** *(radial stretching, an intrinsic feature of the Schwarzschild vacuum geometry).*

A neighbourhood $\mathcal{T} = (r_0 - \delta, r_0 + \delta)$ *containing this zero constitutes the* ***geometric transition zone*** *(reverse-bending zone). Its core geometric feature is that the spacetime curvature smoothly transforms from purely spherical bending induced by matter to the saddle-shaped bending inherent to the vacuum — precisely the mathematical expression of anti-elastic bending in elasticity theory.*

Physical clarification: Although mathematically the reverse-bending zone lies inside the matter boundary, physically it is already a geometrically dominated region; this is precisely the region that traditional post-Newtonian approximations treat as a free vacuum (detailed in Section 4.1).

3 Proof of the Theorem

3.1 Metric, tetrad and sectional curvature

Adopt the most general static spherically symmetric line element (in SI units, explicitly retaining c):

$$ds^2 = -e^{2\Phi(r)}c^2dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{1}$$

where $\Phi(r), \lambda(r) \in C^2[0,\infty)$ are real-valued functions depending only on the radial coordinate r.

Introduce an orthonormal tetrad adapted to the metric:

$$\mathbf{e}_{\hat{t}} = e^{-\Phi(r)}c^{-1}\partial_t, \quad \mathbf{e}_{\hat{r}} = e^{-\lambda(r)}\partial_r, \quad \mathbf{e}_{\hat{\theta}} = r^{-1}\partial_\theta, \quad \mathbf{e}_{\hat{\phi}} = (r\sin\theta)^{-1}\partial_\phi, \tag{2}$$

with the dual basis:

$$\omega^{\hat{t}} = e^{\Phi(r)}c\, dt, \quad \omega^{\hat{r}} = e^{\lambda(r)}dr, \quad \omega^{\hat{\theta}} = r\, d\theta, \quad \omega^{\hat{\phi}} = r\sin\theta\, d\phi. \tag{3}$$

In this frame, the metric reduces to the Minkowski metric: $g = -\omega^{\hat{t}}\otimes\omega^{\hat{t}} + \omega^{\hat{r}}\otimes\omega^{\hat{r}} + \omega^{\hat{\theta}}\otimes\omega^{\hat{\theta}} + \omega^{\hat{\phi}}\otimes\omega^{\hat{\phi}}$.

From the first Cartan structure equation[3, 4] $d\omega^{\hat{\mu}} = -\omega^{\hat{\mu}}{}_{\hat{\nu}} \wedge \omega^{\hat{\nu}}$, the non-vanishing connection 1-forms are:

$$\omega^{\hat{t}}{}_{\hat{r}} = \omega^{\hat{r}}{}_{\hat{t}} = e^{-\lambda}\Phi'\,\omega^{\hat{t}}, \quad \omega^{\hat{\theta}}{}_{\hat{r}} = -\omega^{\hat{r}}{}_{\hat{\theta}} = \frac{e^{-\lambda}}{r}\omega^{\hat{\theta}}, \quad \omega^{\hat{\phi}}{}_{\hat{r}} = -\omega^{\hat{r}}{}_{\hat{\phi}} = \frac{e^{-\lambda}}{r}\omega^{\hat{\phi}}, \quad \omega^{\hat{\phi}}{}_{\hat{\theta}} = -\omega^{\hat{\theta}}{}_{\hat{\phi}} = \frac{\cot\theta}{r}\omega^{\hat{\phi}}. \tag{4}$$

Here and below, a prime denotes differentiation with respect to the radial coordinate r.

From the second Cartan structure equation $\Omega^{\hat{\mu}}{}_{\hat{\nu}} = d\omega^{\hat{\mu}}{}_{\hat{\nu}} + \omega^{\hat{\mu}}{}_{\hat{\lambda}} \wedge \omega^{\hat{\lambda}}{}_{\hat{\nu}}$, the curvature 2-form of the t-r plane is:

$$\Omega^{\hat{t}}{}_{\hat{r}} = d\omega^{\hat{t}}{}_{\hat{r}} + \omega^{\hat{t}}{}_{\hat{\lambda}} \wedge \omega^{\hat{\lambda}}{}_{\hat{r}} = -e^{-2\lambda}\Big[\Phi'' + (\Phi')^2 - \Phi'\lambda'\Big]\,\omega^{\hat{t}} \wedge \omega^{\hat{r}}. \tag{5}$$

The Riemann curvature tensor and the curvature 2-form are related by $\Omega^{\hat{\mu}}{}_{\hat{\nu}} = \frac{1}{2}R^{\hat{\mu}}{}_{\hat{\nu}\hat{\rho}\hat{\sigma}}\,\omega^{\hat{\rho}} \wedge \omega^{\hat{\sigma}}$. Comparing with (5) and using the metric signature convention $g_{\hat{t}\hat{t}} = -1$, we obtain:

$$R_{\hat{t}\hat{r}\hat{t}\hat{r}} = g_{\hat{t}\hat{t}}g_{\hat{r}\hat{r}}R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{r}} = -R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{r}}. \tag{6}$$

Hence, the exact expression for the sectional curvature is:

$$\boxed{K_{tr}(r) \equiv R_{\hat{t}\hat{r}\hat{t}\hat{r}}(r) = -e^{-2\lambda(r)}\Big[\Phi''(r) + \big(\Phi'(r)\big)^2 - \Phi'(r)\lambda'(r)\Big]}. \tag{7}$$

This expression is a purely differential-geometric result that depends solely on the metric functions and holds universally for all static spherically symmetric spacetimes. For the self-gravitating bound system studied in this paper, the matter source is described as an isotropic ideal fluid. In the orthonormal frame (2) adapted to the metric, its energy-momentum tensor takes the diagonal form

$$T_{\hat{\mu}\hat{\nu}} = \mathrm{diag}(\rho c^2,\, p,\, p,\, p), \tag{8}$$

where $\rho(r)$ is the proper energy density and $p(r)$ is the isotropic pressure.

3.2 Exact algebraic identity based on the field equations

In order to rigorously analyse the sign change of $K_{tr}(r)$, we start directly from the Einstein field equations $G_{\mu\nu} + \Lambda g_{\mu\nu} = (8\pi G/c^4)T_{\mu\nu}$ and derive their exact algebraic expression **without neglecting the cosmological constant** Λ.

For an ideal fluid (or effective fluid) matter source, the Einstein equations in the orthonormal frame yield:

$$\frac{2}{r}e^{-2\lambda}\lambda' + \frac{1}{r^2}(1 - e^{-2\lambda}) - \Lambda = \frac{8\pi G}{c^2}\rho, \tag{9}$$

$$\frac{2}{r}e^{-2\lambda}\Phi' - \frac{1}{r^2}(1 - e^{-2\lambda}) + \Lambda = \frac{8\pi G}{c^4}p, \tag{10}$$

$$e^{-2\lambda}\left[\Phi'' + (\Phi')^2 - \Phi'\lambda' + \frac{\Phi' - \lambda'}{r}\right] + \Lambda = \frac{8\pi G}{c^4}p. \tag{11}$$

Define the interior mass $M(r) = \int_0^r 4\pi r'^2 \rho(r')dr'$, which satisfies

$$e^{-2\lambda(r)} = 1 - \frac{2GM(r)}{c^2 r} - \frac{\Lambda}{3}r^2. \tag{12}$$

The geometric definition of the sectional curvature is $K_{tr}(r) \equiv R_{\hat{t}\hat{r}\hat{t}\hat{r}} = -e^{-2\lambda}\big[\Phi'' + (\Phi')^2 - \Phi'\lambda'\big]$. Eliminating the second-derivative term using the angular equation (11), we obtain

$$K_{tr} = -\frac{8\pi G}{c^4}p + \Lambda + \frac{e^{-2\lambda}}{r}(\Phi' - \lambda'). \tag{13}$$

Solving (9) and (10) for λ' and Φ' respectively gives

$$\lambda' = e^{2\lambda}\left[\frac{4\pi G}{c^2}\rho r - \frac{GM}{c^2 r^2} + \frac{\Lambda}{3}r\right], \qquad \Phi' = e^{2\lambda}\left[\frac{GM}{c^2 r^2} + \frac{4\pi G}{c^4}pr - \frac{\Lambda}{3}r\right].$$

Subtracting the two equations yields

$$\Phi' - \lambda' = e^{2\lambda}\left[\frac{2GM}{c^2 r^2} + \frac{4\pi G}{c^4}pr - \frac{4\pi G}{c^2}\rho r - \frac{2\Lambda}{3}r\right].$$

Substituting this back into (13) and simplifying algebraically (the Λ terms cancel exactly) yields the **exact algebraic identity**:

$$\boxed{K_{tr}(r) = \frac{2GM(r)}{c^2 r^3} - \frac{4\pi G}{c^2}\left(\rho(r) + \frac{p(r)}{c^2}\right) + \frac{\Lambda}{3}}. \tag{14}$$

Physical interpretation:

- The first term $\dfrac{2GM(r)}{c^2 r^3}$ represents the non-local contribution from the interior mass; in the vacuum region $r > R$ it is the Weyl stretching (vacuum tide).
- The second term $-\dfrac{4\pi G}{c^2}\left(\rho + \dfrac{p}{c^2}\right)$ represents the Ricci compression due to the local **active gravitational mass density** (note the coefficient 1 in front of the pressure term, consistent with standard general relativity).

- The third term $\frac{\Lambda}{3}$ is a uniform background curvature provided by the cosmological constant. Inside a galaxy (small r) this term is dozens of orders of magnitude smaller than the matter terms and can be completely neglected; however, in the far field it ensures that the curvature tends to the de Sitter constant curvature, which is crucial for the smooth junction between interior and exterior.

Equation (14) is an exact identity that holds **throughout the entire spacetime**: interior, matter boundary, and exterior vacuum are all covered uniformly.

3.3 Sign of the sectional curvature and existence of the zero

From the exact formula (14) we can rigorously determine the sign of $K_{tr}(r)$ at the centre and in the far field, and prove the existence of a zero.

Compression feature at the centre: As $r \to 0$, $M(r) \approx \frac{4}{3}\pi r^3 \rho(0)$, p is finite, and the $\Lambda/3$ term is negligible compared with the matter terms. Substituting into (14):

$$K_{tr}(0) \approx \frac{8\pi G}{3c^2}\rho(0) - \frac{4\pi G}{c^2}\left(\rho(0) + \frac{p(0)}{c^2}\right) = -\frac{4\pi G}{3c^2}\left(\rho(0) + \frac{3p(0)}{c^2}\right) < 0.$$

As long as the strong energy condition holds, the sectional curvature at the centre is strictly negative, corresponding to radial compression.

Positive curvature feature of the exterior vacuum: At the matter boundary $r = R$, the prerequisites give $\rho(R) = p(R) = 0$ and $M(R) = M_{\text{total}} > 0$. Substituting into (14) yields

$$K_{tr}(R) = \frac{2GM_{\text{total}}}{c^2 R^3} + \frac{\Lambda}{3} > 0.$$

Thus, the sectional curvature is already strictly positive at the matter boundary. By Birkhoff's theorem[4] , the exterior vacuum region $r > R$ is uniquely described by the Schwarzschild–de Sitter solution, and $K_{tr}(r) > 0$ is precisely an intrinsic feature of that solution.

Existence and physical location of the zero: Because the matter distribution is C^2 smooth, $M(r)$, $\rho(r)$ and $p(r)$ are continuous functions; therefore $K_{tr}(r)$ is continuous in the whole space. It is negative at $r = 0$ ($K_{tr}(0) < 0$) and positive at $r = R$ ($K_{tr}(R) > 0$). By the Intermediate Value Theorem for continuous functions, there must exist at least one zero $r_0 \in (0, R)$ such that $K_{tr}(r_0) = 0$.

Setting (14) to zero, and using the fact that in the outer part of a galaxy the pressure is negligible ($p \approx 0$) and the $\Lambda/3$ term is far smaller than the matter term on this scale, the zero position is determined by the equation

$$\frac{2GM(r_0)}{c^2 r_0^3} = \frac{4\pi G}{c^2}\rho(r_0) \quad \Longrightarrow \quad \frac{M(r_0)}{\frac{4}{3}\pi r_0^3} = \frac{3}{2}\rho(r_0).$$

Defining the mean mass density inside r_0 as $\bar{\rho}(r_0) = M(r_0)/(\frac{4}{3}\pi r_0^3)$, the zero condition can be written concisely as

$$\boxed{\bar{\rho}(r_0) = 1.5\,\rho(r_0)}. \tag{15}$$

For a three-dimensional spherically symmetric exponential volume density distribution $\rho(r) = \rho_0 e^{-r/h}$[7] ,substituting the cumulative mass $M(r) = 4\pi\rho_0 h^3\big[2 - e^{-r/h}(r^2/h^2 +$

$2r/h+2)\big]$ into $\bar{\rho}(r_0) = 1.5\rho(r_0)$, and letting $x_0 \equiv r_0/h$, elimination of ρ_0 yields the transcendental equation

$$4e^{x_0} = x_0^3 + 2x_0^2 + 4x_0 + 4. \tag{16}$$

Numerical solution gives $x_0 \approx 1.441$, hence

$$\boxed{r_0 \approx 1.44\,h}. \tag{17}$$

This result is derived solely from the zero condition of the sectional curvature in general relativity, without any free parameters.

This relation shows that the reverse-bending curvature zero appears exactly at the radius where "the interior mean density equals 1.5 times the local density". For common density profiles that are single-peaked, monotonically decreasing radially and without oscillatory tails (such as the exponential disc or power-law profiles of galaxies): near the centre $M(r) \approx \frac{4}{3}\pi r^3\rho(0)$, so $\bar{\rho} \approx \rho$, and the ratio $\bar{\rho}/\rho \approx 1 < 1.5$; as r increases, the local density ρ decays faster than the mean density $\bar{\rho}$, so the ratio $\bar{\rho}/\rho$ increases strictly monotonically, and at the matter boundary $r = R$ we have $\rho(R) = 0$ but $\bar{\rho}(R) > 0$, so the ratio diverges to $+\infty$. By the Intermediate Value Theorem, there exists a unique r_0 satisfying $\bar{\rho}(r_0) = 1.5\rho(r_0)$.

It is worth emphasising that the reverse-bending zero r_0 is the starting point of the reverse-bending zone, not the starting point of the Schwarzschild vacuum. In the Schwarzschild vacuum at finite radius the sectional curvature is strictly positive ($K_{tr} = 2GM/(c^2r^3) > 0$), whereas here $K_{tr}(r_0) = 0$. If one tried to match the Schwarzschild solution directly at r_0, the curvature would jump from zero to a positive value, violating the C^2 continuity requirement of the metric.

3.4 Peak of the sectional curvature in the reverse-bending zone

Theorem 2.1 and the exact expression (14) establish that $K_{tr}(r)$ in the reverse-bending zone changes from negative to positive at r_0, remains positive at the matter boundary $r = R$, and in the far field decays to the de Sitter constant curvature background $\Lambda/3$. This means that $K_{tr}(r)$ starts from zero, rises in some interval, reaches a positive maximum, then decreases and eventually tends to the far-field vacuum value.

Under the conditions of Theorem 2.1, let $K_{tr}(r) \in C^1[r_0, \infty)$ satisfy $K_{tr}(r_0) = 0$, $K_{tr}(r) > 0$ for all $r > r_0$, and $\lim_{r\to\infty} K_{tr}(r) = \Lambda/3$. Choose $R_1 > R$ such that for $r > R_1$, $|K_{tr}(r) - \Lambda/3| < \varepsilon$, and take ε sufficiently small so that $\Lambda/3 < K_{tr}(R)$. The continuous function K_{tr} on the closed interval $[r_0, R_1]$ must attain a maximum, and this maximum cannot be at r_0 (because $K_{tr}(r_0) = 0$) nor at infinity (because the limit is $\Lambda/3 < K_{tr}(R)$); therefore there exists at least one interior maximum point $r_{\text{peak}} \in (r_0, R_1)$ such that

$$K_{tr}(r_{\text{peak}}) = \max_{r\geq r_0} K_{tr}(r) > 0, \qquad K'_{tr}(r_{\text{peak}}) = 0.$$

To further confirm that K_{tr} indeed first rises to the right of r_0, one can directly examine the derivative at the zero. Using the approximation $K_{tr} \approx 2GM/(c^2r^3) - 4\pi G\rho/c^2$ in the outer part of the galaxy and the zero condition $\bar{\rho}(r_0) = 1.5\rho(r_0)$, we obtain

$$K'_{tr}(r_0) = -\frac{4\pi G}{c^2}\Big(\frac{\rho(r_0)}{r_0} + \rho'(r_0)\Big).$$

For a radially monotonically decreasing density distribution, $\rho'(r_0) < 0$, and when $r_0 > h$ we have $|\rho'| > \rho/r_0$; thus $K'_{tr}(r_0) > 0$, and the curvature immediately rises on the outer side of the zero. This guarantees the existence of a rising segment, which together with the far-field decay forces at least one interior maximum.

To determine the specific position of the peak, we use the approximation that the pressure in the outer part of the galaxy is negligible ($p \approx 0$) and the $\Lambda/3$ term is far smaller than the matter term on this scale, simplifying the exact expression (14) to

$$K_{tr}(r) \approx \frac{2GM(r)}{c^2 r^3} - \frac{4\pi G}{c^2}\rho(r). \tag{18}$$

Differentiating with respect to r and setting the derivative to zero, using $M'(r) = 4\pi r^2 \rho(r)$, gives the general equation satisfied by the peak position r_{peak}:

$$\boxed{3M(r_{\text{peak}}) = 4\pi r_{\text{peak}}^3\, \rho(r_{\text{peak}}) - 2\pi r_{\text{peak}}^4\, \rho'(r_{\text{peak}})}. \tag{19}$$

This equation contains no free parameters and applies universally to any static spherically symmetric density distribution.

Now take the volume density of the galactic visible matter as a three-dimensional spherically symmetric exponential distribution $\rho(r) = \rho_0 e^{-r/h}$ [7](h being the radial scale length), with cumulative mass

$$M(r) = 4\pi\rho_0 h^3 \left[2 - e^{-r/h}\left(\frac{r^2}{h^2} + \frac{2r}{h} + 2\right)\right].$$

Letting $x \equiv r/h$ and substituting into Eq. (19), after eliminating ρ_0 and simplifying we obtain the transcendental equation for x:

$$\boxed{12e^x = x^4 + 2x^3 + 6x^2 + 12x + 12}. \tag{20}$$

This equation has only one non-trivial real root in the range $x > 0$ (the other root is $x = 0$, corresponding to the degenerate centre solution). Solving numerically with Newton's method yields

$$x_{\text{peak}} \approx 2.8413 \quad \Longrightarrow \quad \boxed{r_{\text{peak}} \approx 2.84\, h}. \tag{21}$$

Non-Schwarzschild matching feature at the peak. The above analysis reveals a crucial geometric fact about the reverse-bending zone: at the curvature peak r_{peak}, $K'_{tr}(r_{\text{peak}}) = 0$. If one attempted to match the interior solution directly to the Schwarzschild vacuum at this point, the exterior curvature $K_{tr}^{\text{Sch}} = 2GM/(c^2 r^3)$ would have the derivative $K_{tr}'^{\text{Sch}} = -6GM/(c^2 r^4) < 0$ (for all finite r), which contradicts the peak condition $K'_{tr}(r_{\text{peak}}) = 0$. From the basic requirement of continuity we have $0 < r_0 < r_{\text{peak}} < R$, i.e. the peak point must lie inside the matter boundary (R). Therefore, under the premises of staticity, spherical symmetry and a C^2 metric, one cannot smoothly join the interior forced geometry to the exterior free Schwarzschild vacuum at r_{peak}. In summary, geometrically the reverse-bending zone is neither purely interior baryon-dominated nor exterior free vacuum; it must complete a smooth transition from growth to decay within a finite extended interval, with a decay rate that is overall slower than $1/r^3$ in order to achieve a smooth connection of the curvature and its derivative. This will be elaborated in Section 4.

3.5 Transition to the exterior Schwarzschild–de Sitter vacuum

Beyond the curvature peak r_{peak}, as the radius continues to increase, the local matter density $\rho(r)$ and pressure $p(r)$ decay exponentially (or faster), and the growth of the cumulative mass $M(r)$ saturates, gradually approaching the total mass of the system M_{total}. From the exact formula (14)

$$K_{tr}(r) = \frac{2GM(r)}{c^2r^3} - \frac{4\pi G}{c^2}\left(\rho(r) + \frac{p(r)}{c^2}\right) + \frac{\Lambda}{3}$$

we see that the second term (local Ricci compression) vanishes together with the matter density, the first term (Weyl stretching) decays as M_{total}/r^3, and the third term $\Lambda/3$ remains as a uniform background curvature. Hence $K_{tr}(r)$ decreases smoothly from the peak and approaches the pure vacuum value as one nears the matter boundary $r = R$.

According to the prerequisites of Theorem 2.1, at the boundary $r = R$ we have

$$\rho(R) = p(R) = 0, \qquad M(R) = M_{\text{total}} > 0.$$

Substituting into (14) immediately gives

$$K_{tr}(R^-) = \frac{2GM_{\text{total}}}{c^2R^3} + \frac{\Lambda}{3}.$$

In the exterior region $r > R$, the matter energy-momentum tensor $T_{\mu\nu} = 0$, and the spacetime satisfies the vacuum Einstein equations with the cosmological constant Λ. By Birkhoff's theorem generalised to include Λ, the static spherically symmetric vacuum solution is uniquely described by the Schwarzschild–de Sitter (Kottler) metric, with the exterior total mass parameter exactly equal to M_{total}, and the t-r sectional curvature is

$$K_{tr}(R^+) = \frac{2GM_{\text{total}}}{c^2R^3} + \frac{\Lambda}{3}.$$

Therefore $K_{tr}(R^-) = K_{tr}(R^+)$, and the sectional curvature is **automatically continuous** at the matter boundary, without the need for any additional matching conditions or thin-shell constructions.

This fact has a fundamental physical significance: Eq. (14), as an exact algebraic identity of the Einstein field equations, already **threads through the entire spacetime**. The interior forced geometry and the exterior free vacuum are not two different solutions that need to be stitched together, but rather continuous manifestations of the same formula on the two sides of the natural matter boundary.

The physical meaning of the boundary R is the dividing line between the interior forced geometry and the exterior free vacuum. At the end of the reverse-bending zone, the exterior vacuum description takes over. Thus, starting from the negative curvature (baryonic compression) at the galactic centre, passing through the curvature zero r_0, the reverse-bending rising segment, the curvature peak r_{peak}, the transition zone, and finally reaching the exterior Schwarzschild–de Sitter vacuum, the evolution of the sectional curvature of the entire spacetime is completely described by the single unified exact formula (14). This closed loop constitutes the rigorous general-relativistic foundation for the geometric explanation of dark matter.

3.6 Geometric transition zone and its physical significance

The above proof using the standard spherically symmetric model establishes that, from the centre to infinity, $K_{tr}(r)$ inevitably undergoes the process "negative (spherical compression) $\to$ crosses zero $\to$ positive (saddle-shaped stretching)". The critical radius r_0 is the exact position of the curvature sign reversal; the interval from there outward up to the matter boundary R, (r_0, R), constitutes the **geometric transition zone** — that is, the **reverse-bending zone**. In this region, $K_{tr}(r) > 0$, and spacetime exhibits saddle-shaped curvature, but this stretching is not the free Schwarzschild stretching of the exterior vacuum at $r > R$; rather, it is a **forced geometry** locked jointly by the interior matter distribution and the far-field asymptotically flat condition.

Hierarchy one: existence proof. The spherically symmetric geometry acts as a "detection tool". It uses a known, rigorous geometric structure to prove that, for a given matter distribution and vacuum boundary conditions, a curvature zero necessarily exists inside. This step requires no new physics and is entirely an internal deduction of standard general relativity.

Hierarchy two: properties of the reverse-bending zone. With the zero proven to exist, the reverse-bending zone is established as a geometric entity. However, whether the metric form and gravitational effects in this zone still obey the logic of the standard vacuum solution is a separate question. The Schwarzschild metric proves that r_0 necessarily exists and $r_0 < R$, but that does not mean the Schwarzschild metric correctly describes the reverse-bending zone. Although the reverse-bending zone is not a vacuum ($\rho \neq 0$), the local matter density is already extremely low, its Ricci compression contribution is negligible, and the geometric effect is completely dominated by the non-local Weyl stretching produced by the interior mass. (A detailed argument is given in Section 4.)

On the universality of the reverse-bending zone: The five conditions on which the proof of Theorem 2.1 depends form a set of sufficient conditions for the existence of the reverse-bending zone, not necessary conditions. The spherically symmetric results show that the reverse-bending mechanism does not rely on any special microscopic assumptions, but stems from the combined action of a finite matter distribution, the exterior vacuum boundary, and geometric regularity. This suggests that the mechanism can be extended to more general self-gravitating systems, although its rigorous non-spherically symmetric form awaits future investigation.

The role of the spherical symmetry assumption: It is not a prerequisite for the existence of the reverse-bending zone, but rather a way to convert the general physical conditions into an exactly solvable mathematical model. Under spherical symmetry, the sectional curvature K_{tr} happens to correspond to an eigenvalue of the tidal force, and the position of its zero can be analytically determined as $\bar{\rho}(r_0) = 1.5\,\rho(r_0)$. For realistic non-spherically symmetric galaxies, the reverse-bending zone still exists, but its precise geometric form requires solving the corresponding non-linear boundary value problem, which will be a direction for future numerical work. Spherical symmetry in general relativity has never been an exact description of reality, but a standard methodological tool for uncovering universal physical mechanisms. All cornerstone results of general relativity — Birkhoff's theorem, the Schwarzschild solution, the Oppenheimer-Volkoff equation, Friedmann cosmology, etc. — are built on the assumption of spherical symmetry, yet this in no way diminishes their universality.

4 Forced vacuum geometry and the emergence of dark-matter effects

The geometric transition theorem of Section 3 rigorously proves that a geometric transition zone (reverse-bending zone) with curvature sign reversal exists in the periphery of the visible matter, with the zero r_0 located at $\bar{\rho}(r_0) = 1.5\,\rho(r_0)$. In this section we demonstrate that, once this inner zero-curvature boundary is crossed, the spacetime metric enters a forced geometric state constrained jointly by the interior integrated mass and the exterior boundary. In its weakest form this state causes the effective mass parameter in the tail to deviate from the bare baryonic extrapolation, and when further satisfying a self-similarity condition in the mature zone it can evolve into a scale-invariant logarithmic potential zone, giving rise to flat rotation curves and the associated scaling relations.

4.1 Forced geometry: the range of applicability of the post-Newtonian approximation and Birkhoff's theorem

In astrophysical research, when analysing gravitational effects in the outer parts of galaxies, the post-Newtonian approximation is commonly adopted, and it is assumed by default that spacetime reduces to the Schwarzschild vacuum described by Birkhoff's theorem.

According to the spherically symmetric derivation, in the reverse-bending region $0 < r_0 < r_{\text{peak}} < R$, the curvature zero appears exactly at the radius where "the interior mean density equals 1.5 times the local density", and the peak r_{peak} lies inside the matter boundary R. This result indicates that the low-density outer part of a galaxy belongs geometrically to a forced transition zone: its spacetime metric is neither dominated solely by the local tenuous matter, nor does it obey the free Schwarzschild vacuum form, but is constrained jointly by the interior baryonic potential well and the far-field boundary condition. In this region the local matter density can be extremely low, yet its spacetime curvature and the non-linear geometric self-energy contributions may not be negligible.

Birkhoff's theorem is strictly valid in spherically symmetric source-free regions where the matter density is exactly zero. In contrast, the dynamical information from the outer part of a galaxy comes from tracer objects embedded in that region; as far as the system attribution is concerned, this region still belongs to the self-gravitating matter envelope, rather than being an independent vacuum exterior completely detached from the central matter. Therefore, treating it entirely as a strictly source-free region in the sense of Birkhoff's theorem can at best be regarded as an approximate treatment. Especially in the reverse-bending zone, where the curvature has already undergone a reorganisation from negative to positive sign, the geometric structure is still controlled by the interior mass distribution and non-linear boundary conditions. Hence, Birkhoff's theorem applies to the strictly vacuum exterior region outside the reverse-bending zone, but cannot serve as a direct substitute for the geometric structure inside the reverse-bending transition layer.

4.2 Physical effects of the forced geometry in the reverse-bending zone: the emergence of dark-matter phenomena

4.2.1 Dominance of the Weyl tensor and the self-energy of the gravitational field

The most central non-linear essence of general relativity is: **the gravitational field itself carries energy, and the energy of the gravitational field also produces gravity**.

We have already shown above that the reverse-bending zone is geometrically constrained by three points — the zero r_0, the peak r_{peak}, and the matter boundary R — thereby forming a forced geometric transition layer. Even under the most conservative treatment — matching to the Schwarzschild vacuum immediately after the curvature reaches its positive peak r_{peak} — the first half of the reverse-bending enhancement zone is ineliminable. In this zone the local baryon density has already been continuously decreasing, yet the radial sectional curvature K_{tr} rises from zero to its maximum positive value $K_{\max}$, which means that the gravitational behaviour here is no longer controlled by the local Ricci source term alone, but must be understood as the joint result of the overall integrated mass, the boundary conditions, and geometric non-linearity. Therefore, even this first half of the reverse-bending alone is sufficient to produce additional gravitational effects that deviate from the Newtonian expectation.

According to the expression for the sectional curvature (14)

$$K_{tr}(r) = \frac{2GM(r)}{c^2 r^3} - \frac{4\pi G}{c^2}\left(\rho(r) + \frac{p(r)}{c^2}\right) + \frac{\Lambda}{3},$$

when matched to the Schwarzschild-type tail after $r = r_{\text{peak}}$, the mass parameter of the latter must satisfy

$$K_{\max} = \frac{2GM_{\text{peak}}}{c^2 r_{\text{peak}}^3} \quad \Rightarrow \quad M_{\text{peak}} = \frac{c^2 r_{\text{peak}}^3}{2G} K_{\max}, \tag{22}$$

yielding the conclusion that even if the tail still has the form $K_{tr} \propto r^{-3}$, the "mass parameter" M_{peak} that it decays from is no longer determined solely by the bare baryonic mass inside r_{peak}, but is determined jointly by the curvature peak built up in the first half of the reverse-bending enhancement zone through the matching condition. In dynamical inversion, this manifests itself as an effective mass parameter that may deviate significantly from the extrapolated bare baryonic value. This conclusion relies on staticity, spherical symmetry and the weak-field approximation, but does not depend on whether there exists a long-range logarithmic tail beyond the peak, thus constituting a robust conclusion under minimal assumptions.

4.2.2 Definition and qualitative analysis of the total mass in the reverse-bending zone

The $M(r)$ defined in Eq. (12) of this paper is fully consistent with the standard *Misner–Sharp* quasi-local mass definition in static spherically symmetric spacetime. Because this term is mathematically exactly equivalent to the standard Misner–Sharp mass, it completely captures the contributions of all forms of gravitational sources within radius r, a conclusion widely accepted in the physics community. In view of the fact that Eq. (14)

exhibits an exact algebraic coupling between $M(r)$ and the sectional curvature K_{tr}, in the reverse-bending zone the growth of $M(r)$ no longer follows the linear extrapolation of the bare baryonic matter, but is an inevitable consequence of the non-linear superposition of the baryon distribution and the geometric self-energy of the gravitational field. To facilitate distinction, we give it a new name in the reverse-bending zone, defining it as the "gravitational mass". The essence behind this name remains unchanged, and is meant to emphasise that, in the forced geometric transition zone, the depth of the gravitational potential well is supported jointly by matter and spacetime curvature. According to the exact algebraic identity (14) from Section 3.2:

$$K_{tr}(r) = \frac{2GM(r)}{c^2 r^3} - \frac{4\pi G}{c^2}\left(\rho(r) + \frac{p(r)}{c^2}\right) + \frac{\Lambda}{3}.$$

Solving for the non-local mass term $M(r)$, we define the gravitational mass as:

$$\boxed{M_{\text{geom}}(r) \equiv M(r) \equiv \frac{c^2 r^3}{2G}\left[K_{tr}(r) + \frac{4\pi G}{c^2}\left(\rho(r) + \frac{p(r)}{c^2}\right) - \frac{\Lambda}{3}\right]}. \tag{23}$$

In (23), $M_{\text{geom}}(r)$ is the total mass including the baryonic contribution and the geometric self-energy contribution. This is a strict algebraic identity that does not depend on any approximation. The evolution of the gravitational mass can be read off directly. In the reverse-bending zone ($r_0 < r < R$), the local density and pressure are strictly positive, the sectional curvature is strictly positive, and the cosmological constant term can be neglected.

Rapid growth segment ($r_0 \to r_{\text{peak}}$): K_{tr} rises from zero to the maximum $K_{\max}$, while $\rho + p/c^2$ decreases but still contributes positively to the growth. Superposing the two, the expression inside the brackets increases as a whole, and multiplying by the explicit factor r^3 from the prefactor $c^2 r^3/(2G)$ — this makes the gravitational mass grow rapidly. This is the most drastic phase of gravitational mass growth.

Decelerating growth segment ($r_{\text{peak}} \to R$): After passing the peak, K_{tr} decreases from the maximum but remains positive; $\rho + p/c^2$ continues to decrease but still contributes positive growth. The change of the gravitational mass here depends on the competition between the rate of decline of K_{tr} and the growth rate of the r^3 factor in the prefactor $c^2 r^3/(2G)$.

Note: ($R < r$) is already outside the reverse-bending zone, the Schwarzschild decay segment governed by Birkhoff's theorem.

4.2.3 Quantitative proof of the persistent growth of the gravitational mass in the reverse-bending zone

(1) Rapid growth segment ($r_0 \to r_{\text{peak}}$) — the gravitational mass grows faster than linearly

In the first half of the reverse-bending zone $r_0 < r < r_{\text{peak}}$, the gravitational mass $M(r)$ grows faster than linearly. This conclusion can be verified by exact calculation for the exponential density distribution. Introduce the dimensionless radial coordinate $x \equiv r/h$ and the normalised mass function

$$S(x) \equiv \frac{M(r)}{4\pi \rho_0 h^3} = 2 - e^{-x}(x^2 + 2x + 2), \tag{24}$$

where ρ_0 is the central density and h is the radial scale length. The curvature zero x_0 is determined by the condition $\overline{\rho}(r_0) = 1.5\,\rho(r_0)$ as $x_0 \approx 1.441$ (see (17)), and the curvature peak x_{peak} is given by the peak equation (20) with the numerical solution $x_{\text{peak}} \approx 2.841$.

At these two positions the normalised masses are

$$S(x_0) \approx 0.353, \qquad S(x_{\text{peak}}) \approx 1.080.$$

Now construct a linear-growth reference curve starting from the zero:

$$S_{\text{lin}}(x) \equiv S(x_0)\,\frac{x}{x_0}.$$

This curve describes the case where the mass grows strictly as $M(r) \propto r$. At the peak, the mass extrapolated linearly is

$$S_{\text{lin}}(x_{\text{peak}}) = 0.353 \times \frac{2.841}{1.441} \approx 0.696.$$

The ratio of the actual mass to the linear extrapolation is

$$\frac{S(x_{\text{peak}})}{S_{\text{lin}}(x_{\text{peak}})} \approx \frac{1.080}{0.696} \approx 1.55.$$

This ratio is significantly larger than 1, proving that in the interval $r_0 \to r_{\text{peak}}$ the growth rate of the total mass $M(r)$ is strictly faster than linear. From a differential viewpoint, the mass growth in this interval satisfies

$$\frac{dM}{dr} > \frac{M}{r},$$

i.e. the local mass increment exceeds the linear growth required by a uniform distribution. The fundamental reason is that the curvature K_{tr} rapidly climbs from zero to its maximum $K_{\max}$, driven jointly with the gradually decaying but still positive local matter term $\rho + p/c^2$, and further amplified by the r^3 volume factor. The mass reaching 1.55 times the linear extrapolation implies that the circular orbital rotation speed $v_{\text{rot}} \propto \sqrt{M/r}$ at the end of this interval is enhanced by a factor of about $\sqrt{1.55} \approx 1.245$, i.e., relative to the case of pure linear mass growth, the rotation speed increases by an additional $\sim 24.5\%$. This magnitude is sufficiently large to be resolved by current galaxy rotation-curve observations, providing a clear numerical basis for testing the theory.

(2) Decelerating growth segment ($r_{\text{peak}} \to R$) — the growth rate of the gravitational mass slows down but the mass still continues to increase

According to the three-stage analysis of mass evolution in Section 4.2.2, in the stage $r_{\text{peak}} \to R$ the competition between the growth rates of the sectional curvature and the r^3 factor is key. Here we quantify this rigorously. Differentiating the exact identity (14) directly and substituting $M'(r) = 4\pi r^2 \rho(r)$ gives the exact curvature evolution equation

$$K'_{tr} = -\frac{3}{r}K_{tr} - \frac{4\pi G}{c^2 r}\left(\rho + \frac{3p}{c^2}\right) - \frac{4\pi G}{c^2}\left(\rho' + \frac{p'}{c^2}\right) + \frac{\Lambda}{r}. \tag{25}$$

In the reverse-bending zone $r_0 < r < R$, we have $\rho > 0,\ p > 0,\ \rho' < 0,\ p' < 0$, and Λ/r is negligible on galactic scales. Analysing the signs of the various contributions on the right-hand side:

- The second term $-\frac{4\pi G}{c^2 r}\left(\rho + \frac{3p}{c^2}\right)$ is a negative contribution (since $\rho > 0,\ p > 0$);
- The third term $-\frac{4\pi G}{c^2}\rho'$ is a **strictly positive** contribution (since $\rho' < 0$);
- The fourth term $-\frac{4\pi G}{c^2}\frac{p'}{c^2}$ is a **strictly positive** contribution (since $p' < 0$).

For the common exponentially decaying density profile $\rho \propto e^{-r/h}$, we have $\rho' = -\rho/h$. Neglecting the pressure terms, the positive contribution is $\frac{4\pi G}{c^2}\frac{\rho}{h}$ and the absolute value of the negative contribution is $\frac{4\pi G}{c^2}\frac{\rho}{r}$. When $r > h$ the positive contribution strictly exceeds the negative contribution. Inside the reverse-bending zone $r > r_0 \approx 1.44h > h$, this condition is automatically satisfied. For general single-peaked monotonically decaying density profiles, in the region $r > h$ we also have $|\rho'| > \rho/r$, and the conclusion that the positive contributions dominate still holds. Therefore, the overall net contribution of the last three terms on the right-hand side is strictly positive.

Dropping the net positive contribution, we obtain a strict inequality for the curvature decay:

$$\boxed{K'_{tr} > -\frac{3}{r}K_{tr}}. \tag{26}$$

Physical meaning: The persistent presence of matter provides support to the sectional curvature, making its decay rate strictly slower than the Schwarzschild vacuum law $K'_{tr} = -3K_{tr}/r$.

Derivation of a lower bound for the mass: Now connect this curvature inequality with the mass-curvature correspondence in the outer region. In the outer low-density region $(\rho, p/c^2 \to 0)$, the definition of the gravitational mass (23) reduces to

$$M_{\text{geom}}(r) \propto r^3 K_{tr}(r).$$

Differentiating gives the mass change rate:

$$M'_{\text{geom}} \propto 3r^2 K_{tr} + r^3 K'_{tr} = r^2 K_{tr}\left(3 + \frac{rK'_{tr}}{K_{tr}}\right).$$

Substituting the rigorous conclusion (26) for the curvature evolution, since $K'_{tr} > -3K_{tr}/r$, the term in parentheses is strictly positive, and therefore

$$\boxed{M'_{\text{geom}} > 0}.$$

Throughout the second half of the reverse-bending zone, the curvature decay is necessarily slower than that of the Schwarzschild vacuum, and the gravitational mass continues to increase, its growth rate merely slowing down as the curvature decreases.

(3) Summary

In this section we have proved and quantified, through a combination of analytic and numerical methods, the behaviour of the "reverse-bending zone" for a static spherically symmetric exponential density distribution: in the first half, from the curvature zero r_0 to the peak r_{peak}, the growth rate of the gravitational mass $M(r)$ is significantly faster than a simple linear extrapolation; after crossing the peak, although the growth rate begins to slow down, the effective mass still continues to increase all the way to the matter boundary R.

The reverse-bending zone is not a simple transition layer, but a non-linear gravitational enhancement zone locked jointly by the baryon distribution, the boundary conditions, and the geometric self-energy. Without introducing any new particle hypothesis, it naturally induces extra gravity that deviates from the Newtonian expectation. This growth of the gravitational mass can dynamically manifest itself as the flattening of rotation curves, and geometrically ensures the continuity of the metric at the matter boundary.

In summary, proving that the gravitational mass grows inside the reverse-bending zone proves that, relative to the expectation from the bare baryon distribution, additional gravitational effects necessarily exist. This provides a rigorous general-relativistic foundation for understanding the gravitational anomalies on galactic scales.

4.3 Logarithmic potential as an effective approximate description of the forced geometry

Because in the first half of the reverse-bending zone ($r_0 < r < r_{\text{peak}}$) the mass grows faster than linearly, while in the second half ($r_{\text{peak}} < r < R$) the growth continues but at a decelerating rate, and eventually outside the matter boundary it transitions to the constant-mass Schwarzschild tail, there necessarily exists a natural intermediate window $[r_1, r_2] \subset (r_{\text{peak}}, R)$ in the second half such that the total gravitational mass approximately satisfies $M(r) \propto r$. Within this window, the gravitational potential exhibits approximately logarithmic growth and the rotation curve is approximately flat.

To quantitatively characterise the evolution pattern of the mass growth, introduce the local power-law index

$$\gamma(r) \equiv \frac{d \ln M}{d \ln r}.$$

The physical meaning of $\gamma(r)$ is the instantaneous power-law dependence of the mass growth on radius: if $M(r) \propto r^{\gamma}$, then $d \ln M / d \ln r = \gamma$. From the quantification in Section 4.2.3, we have proved that in the whole reverse-bending zone:

- In $r_0 < r < r_{\text{peak}}$ (first half), $\gamma(r) > 1$, the mass grows super-linearly, and the rotation speed continues to rise;
- At $r = r_{\text{peak}}$, $\gamma(r)$ starts to decrease from a value greater than 1;
- In $r_{\text{peak}} < r < R$ (second half), $\gamma(r)$ remains positive but gradually decreases, the mass still grows, albeit with a decelerating rate.

Combining the complete boundary conditions — $\gamma > 1$ in the first half, $\gamma > 0$ and decreasing in the second half, and $\gamma \to 0$ outside the matter boundary (the constant-mass Schwarzschild tail) — one concludes that in the second half there must exist an intermediate window $[r_1, r_2] \subset (r_{\text{peak}}, R)$ where

$$\gamma(r) \approx 1.$$

Within this window, the gravitational mass approximately satisfies linear growth

$$M(r) \approx \mu r, \qquad \mu = \text{constant}.$$

From the circular orbital velocity formula, we obtain

$$v_{\text{rot}}(r) \approx \sqrt{\frac{G\mu}{r} \cdot r} = \text{constant}, \tag{27}$$

and the gravitational potential is approximately logarithmic:

$$\Phi(r) \sim G\mu \ln r. \tag{28}$$

Thus, in this paper the "logarithmic potential" should be understood as an **effective approximate description** of a small segment in the second half of the reverse-bending zone, not as a strict exact solution over the whole interval.

5 Falsifiable predictions and outlook

5.1 Falsifiable predictions

This theory yields the following rigorous, falsifiable predictions that can be clearly distinguished from the particle dark-matter paradigm:

1. **Geometric locking of the starting position of the reverse-bending zone.** The zero condition of Theorem 2.1, $\bar{\rho}(r_0) = 1.5\rho(r_0)$, locks the starting radius of the reverse-bending zone r_0 into a definite ratio with the disc scale length h of the galaxy's visible matter. For a three-dimensional spherically symmetric exponential density distribution, the strict solution is $r_0 \approx 1.44\,h$ (corresponding to the volume density form); for the surface density form of a thin exponential disc, the corresponding value of r_0/h can be derived separately by integrating the surface density. This prediction contains no free parameters and can be tested directly by statistical comparison of large-sample galaxy photometric data (yielding h) with characteristic radii of rotation curves.

2. **Deterministic relationship between the effective dark-matter profile and the visible matter distribution.** The geometric effective mass distribution of the reverse-bending zone $0 < r_0 < r_{\text{peak}} < R$ is uniquely determined by the visible matter density profile via the Einstein field equations. The overall position and shape of the reverse-bending zone and the total mass growth relationship are essentially determined, and can be tested against observations. This is the fundamental difference between this theory and particle dark-matter models.

3. **Positive correlation between the strength of the reverse-bending zone and the compactness of the system.** The non-local term $2GM/(c^2r^3)$ in Eq. (14) endows the sectional curvature with sensitivity to the scale of the system. For galaxies of the same total mass but smaller scale length h, r_0 is smaller, the curvature peak $K_{\max}$ is larger, and the geometric enhancement of the reverse-bending zone is more pronounced. Hence the theory predicts that the ratio of dynamical mass to baryonic mass, $M_{\text{dyn}}/M_{\text{bar}}$, should be positively correlated with the compactness $1/r_0$ of the system. Compact systems such as dwarf spheroidal galaxies should exhibit systematically larger mass discrepancies, a trend that can be tested with existing survey data.

4. **Correspondence of characteristic points in the rotation-curve morphology.** The theory predicts that the rotation curve undergoes a curvature sign reversal near r_0, and the sectional curvature reaches its maximum at the curvature peak $r_{\text{peak}} \approx 2.84\,h$. In this interval the total mass grows super-linearly, so it is predicted that this interval corresponds to a rising segment of the velocity curve.

These features can be directly compared with the characteristic points (such as inflection points, flat-starting points) of observed rotation curves, constituting a parameter-free morphological test.

5.2 Conclusions and outlook

We have derived the reverse-bending zone entirely within the framework of standard general relativity, providing a geometric mechanistic understanding for the phenomena traditionally attributed to dark matter: the non-local Weyl stretching brought about by the reversal of the curvature sign, together with the non-linear effect of the self-energy of the gravitational field, jointly contribute an additional centripetal gravity, which dynamically manifests itself as effective dark matter. Under the forced-geometry condition, the mass grows continuously, and in a local window it approximates linear growth, thereby manifesting as approximately flat rotation curves. The zero condition and the peak condition yield quantitative, parameter-free predictions that can be statistically tested with large-scale galaxy surveys, offering a clear contrast with the particle dark-matter paradigm.

The present framework is currently built upon the assumption of static spherical symmetry. Future work can proceed along the following directions:

(1) Generalise the spherical-symmetry theorem to axisymmetric and generally static non-spherically symmetric configurations, in order to describe the structure of real spiral galaxies more precisely;

(2) Study the formation and evolution of the reverse-bending zone in dynamical spacetimes, and explore its possible role on cosmological scales;

(3) Use large-sample rotation-curve databases such as SPARC and weak-lensing survey data to carry out systematic tests of the quantitative predictions made in this paper.

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