# Elementary Weak Solutions of the Navier–Stokes Equations and an Application of the Banach Fixed-Point Theorem

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**Abstract**

This is an application of elementary functional analysis in the sense that there are no long or complicated calculations, and the theory of evolution equations is not used at all.

## 1 Introduction

The existence of solutions is actually known. For example, Fujita-Kato Theory, Shibata Theory: Kato [4], Zhang [12], Charve-Danchin [2], Shibata-Murata [6]. The semi-group theory or a priori estimates are in these theories, but these are not elementary. Semi-group theory or a priori estimates for contraction mappings are not used in the proof of the existence of Leray-Hopf's weak solutions (for example, Wasao Sibagaki and Hisako Rikimaru [7]), but they are not elementary either. We define new weak solutions with uniqueness and smoothness, without semi-group theory or a priori estimates. We apply the local solvability of the partial differential operators with constant coefficients. The policy is to let $L$ be the heat operator $\partial_t - \Delta$ in the initial value problem of the Navier–Stokes equations in $\Omega$

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = f - \nabla \mathfrak{p} - (u \cdot \nabla)u, \\ \operatorname{div}(u) = 0, \\ u(0,x) = a(x) \quad \text{for} \quad (0,x) \in \Omega. \end{cases}$$

The uniqueness follows without boundary conditions.

There are no long or complicated calculations; semi-groups, a priori estimates, or boundary conditions on $\Omega' \subset \mathbb{R}^3$ are not used at all. We apply the local solvability of linear partial differential operators with constant coefficients ([8], [3]), erase the pressure $\mathfrak{p}$ by the Helmholtz projection $P$, and solve

$$\frac{\partial u}{\partial t} - \Delta u = Pf - P((u_n \cdot \nabla)u_n),$$
$$\frac{\partial u}{\partial t} - \Delta u = Pf - P((u \cdot \nabla)u),$$

in $X = \bigcap_{m\geq 5,\, p=1,2} W^{m,p}(\Omega)$, $\Omega \subset \mathbb{R} \times \mathbb{R}^3$ is a bounded domain with Lipschitz boundary such that $(0,0) \in \Omega$.

For example, cylinder sets $\Omega = I \times \Omega' = (-T,T) \times B_3(0,R)$. We handle the variables $t, x$ simultaneously. Cylinder sets are Lipschitz domains, so we can use the Sobolev embedding theorem in $\Omega$ and Helmholtz decomposition in $\Omega'$.

## 2 Definition

For convenience, we write the index of the component of the vectors in the upper right corner.

"Function space" and "space" are abbreviations for "linear topological space" (of functions or distributions), other than pressure $\mathfrak{p}$, which are $\mathbb{R}$-valued. The absolute value of the functions in the norm of the normal function space is interpreted as the length of the number vector (the absolute value of $\mathbb{R}^3$ in the norm of the space of the $\mathbb{R}$-valued functions). We write the space of the real numeric functions and the space of the $\mathbb{R}^3$-valued functions using the same symbol to keep the notation simple. Let $|\Omega|$ be the Lebesgue measure of $\Omega$. Let $\chi_\Omega$ be the characteristic function on $\Omega$. For any natural number $m > \max\{0 + 4/1, 0 + 4/2\} = 4, p = 1, 2$, We define the function spaces on $\mathbb{R} \times \mathbb{R}^3$:

$$V^{m,p}(\Omega) = \left\{ u = (u^1, u^2, u^3) : u^i \in C^\infty(\Omega), \left\| u^i \right\|_{W^{m,p}(\Omega)} < \infty \right\},$$
$$V_\sigma^{m,p}(\Omega) = \{ u \in V^{m,p}(\Omega) : \operatorname{div}(u) = 0 \}.$$

Let $W^{m,p}(\Omega)$ and $W_\sigma^{m,p}(\Omega)$ be the Sobolev spaces defined by the completions of $V^{m,p}(\Omega)$ and $V_\sigma^{m,p}(\Omega)$ (cf. [11]).

Let $\mathcal{D}(\Omega)$ be the space of the test functions ($C_0^\infty(\Omega)$ as a set); let $\mathcal{D}_\sigma(\Omega)$ be the space of the test functions for which the divergence is 0 with respect to spatial variables. Let $P\colon L^2(\Omega) \to L_\sigma^2(\Omega) = \overline{\mathcal{D}_\sigma(\Omega)}^{\|\cdot\|_{L^2(\Omega)}}$ be the projection. Let $C^{k,\varepsilon}(\Omega)$ be the Hölder space. Let

$$\begin{aligned}\langle w, \varphi \rangle :=& (w, \varphi)_{L^2(\Omega)} \\ =& \int_\Omega w(t,x) \cdot \varphi(t,x)\, dt\, dx \\ =& \int_\Omega \sum_{i=1}^{3} w^i(t,x) \varphi^i(t,x)\, dt\, dx\end{aligned}$$

for $w = (w^1, w^2, w^3), \varphi = (\varphi^1, \varphi^2, \varphi^3)$.

In general, if for two Banach spaces $X, Y$, there exists a linear Hausdorff space $Z$ such that $X, Y \subset Z$, then $X \cap Y$ is a Banach space with norms given by $\|u\|_X + \|u\|_Y$ or $\max\{\|u\|_X, \|u\|_Y\}$.

$$\max\{\|u\|_X, \|u\|_Y\} \leq \|u\|_X + \|u\|_Y \leq 2\max\{\|u\|_X, \|u\|_Y\}$$

so these are equivalent. We put

$$\mathcal{X} = \bigcap_{m \geq 5} W^{m,1}_\sigma(\Omega) \cap W^{m,2}_\sigma(\Omega)$$
$$\mathcal{X}' = \bigcap_{m \geq 5} W^{m,1}(\Omega) \cap W^{m,2}(\Omega)$$

We define for any $u \in \mathcal{X}$,

$$\|u\|_X = \sum_{m=5}^{\infty} \sum_{k=0}^{\infty} \frac{1}{m!^4 (c'_{k,m})^2} \|u\|_{W^{m,1}(\Omega) \cap W^{m,2}(\Omega)},$$

and for any $u \in \mathcal{X}'$,

$$\|u\|_{X'} = \sum_{m=5}^{\infty} \sum_{k=0}^{\infty} \frac{1}{m!^4 (c'_{k,m})^2} \|u\|_{W^{m,1}(\Omega) \cap W^{m,2}(\Omega)}.$$

We put $X = \{u \in \mathcal{X} : \|u\|_X < \infty\}, X' = \{u \in \mathcal{X}' : \|u\|_{X'} < \infty\}$. For constants $M, C > 0$, let $S$ be a subset of $X$ :

$$S = \{u \in X : \|u\|_X \leq M, \|\partial_{x^j} u\|_X \leq C\|u\|_X\}.$$

Let $E$ be the fundamental solution of $\partial_t - \Delta$. That is, in the sense of that $\mathbb{R}^3$-valued distribution,

$$(\partial_t - \Delta)E(t, x) = \delta(t, x) = \delta(t) \otimes \delta(x).$$

Here,

$$E(t, x) = \begin{cases} \frac{1}{\sqrt{4\pi t}^3} \exp\left(-\frac{|x|^2}{4t}\right) & (t > 0) \\ 0 & (t \leq 0) \end{cases}.$$

(cf. [8], [3]).

## 3 Existence of elementary weak solutions

**Assumption.** Let $\Omega \subset \mathbb{R} \times \mathbb{R}^3$ be a bounded domain with Lipschitz boundary such that $(0, 0) \in \Omega$. Let $f : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ be an external force satisfying

$Pf \in S$. When $f \neq 0$, we further assume that there exists $(0,x) \in \Omega$ such that

$$\int_{\mathbb{R}\times\mathbb{R}^3} E(s,y)(\chi_\Omega Pf)(0-s,x-y)\,ds\,dy \neq 0.$$

Let $A$ denote the set of initial values given by

$$A = \left\{ u(0,\cdot) : u \in X, u(t,x) = \int_{\mathbb{R}\times\mathbb{R}^3} E(s,y)(\chi_\Omega P(f-(u\cdot\nabla)u))(t-s,x-y)\,ds\,dy \right\}.$$

**Proposition 1** (Local solvability of linear partial differential operator with constant coefficients)**.** Let $L$ be a linear partial differential operator with constant coefficients in $\mathbb{R}^N$, and let $E \in \mathcal{D}'(\mathbb{R}^N)$ be a fundamental solution of $L$, i.e., $LE = \delta$. For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and a bounded domain $\Omega \subset \mathbb{R}^N$, a weak solution $u \in \mathcal{D}'(\Omega)$ to the equation $Lu = f$ in $\Omega$ is given by $u = E * \chi_\Omega f \in \mathcal{D}'(\Omega)$.

*Proof.* For any $\varphi \in \mathcal{D}(\Omega)$,

$$\begin{aligned}
\langle L(E*\chi_\Omega f), \varphi\rangle &= \pm\langle E*\chi_\Omega f, L\varphi\rangle \\
&:= \pm\langle E(x), \langle \chi_\Omega(y)f(y), L\varphi(x+y)\rangle\rangle \\
&= \pm\langle \chi_\Omega(x)f(x), \langle E(y), L\varphi(x+y)\rangle\rangle \\
&= \langle \chi_\Omega(x)f(x), \langle LE(y), \varphi(x+y)\rangle\rangle \\
&= \langle LE(x), \langle \chi_\Omega(y)f(y), \varphi(x+y)\rangle\rangle \\
&= \langle \chi_\Omega(y)f(y), \varphi(y)\rangle \\
&= \langle f, \varphi\rangle
\end{aligned}$$

□

**Theorem 1** (Main theorem)**.** $A \neq \emptyset$. For $a \in A$, there are elementary weak solutions $u, \mathfrak{p}$ of the initial value problem in $\Omega$ :

$$\begin{aligned}
\frac{\partial u}{\partial t} - \Delta u &= f - \nabla\mathfrak{p} - (u\cdot\nabla)u, \\
\mathrm{div}(u) &= 0, \\
u(0,x) &= a(x).
\end{aligned}$$

in the sense that $u \in X, \mathfrak{p} \in L^2_{\mathrm{loc}}(\Omega)/(p \sim q \iff \nabla(p-q) = 0)$ and $u, \mathfrak{p}$ satisfy for any $\varphi \in \mathcal{D}_\sigma(\Omega)$,

$$\langle \partial_t u + (u\cdot\nabla)u - \Delta u + \nabla\mathfrak{p} - f, \varphi\rangle = 0,$$

for any $\varphi \in \mathcal{D}(\Omega)$,

$$\langle \mathrm{div}(u), \varphi\rangle = -\sum_{j=1}^{3} \left\langle u^j, \partial_{x^j}\varphi\right\rangle = 0.$$

For any multi-index $\alpha \in \mathbb{Z}^4_{\geq 0}$,

$$\lim_{t,|x|\to\infty} \partial^\alpha u(t,x) = 0.$$

If solutions $u, \mathfrak{p}$ and $v, \mathfrak{q}$ satisfy $u, v \in S, u - v \in S$, then $u = v, \mathfrak{p} = \mathfrak{q}$ on $\Omega$. The map $f \mapsto u$ is continuous.

We proof Theorem 1 later. The method is similar to the proof of Banach's fixed point theorem. $f$ defines $A$ and the proof of existence does not depend on $A$.

**Proposition 2** (Smoothness of elementary weak solutions)**.** The solutions $(u, \mathfrak{p})$ are $C^\infty$-functions.

*Proof.* $m \in \mathbb{N}$ can be taken arbitrarily large, so the embedding theorem into Hölder space ([1]),

> If $\mathbb{N} \ni m - 4/p > 0$, then $W^{m,p}(\Omega) \subset C^{(m-4/p)-1,\varepsilon}(\Omega)$ for all $\varepsilon \in (0, 1)$.

in the sense of existence of suitable representative elements, $u$ is a $C^\infty$-function.

$f$ is smooth and $\partial_t u + (u \cdot \nabla)u - \Delta u - f = -\nabla \mathfrak{p}$. Because $-\nabla \mathfrak{p}$ is smooth, so $\mathfrak{p}$ is also smooth. □

**Lemma 1.** The normed spaces $X$ and $X'$ are non-trivial.

*Proof.* It suffices to exhibit non-zero elements in each space. We note that $\chi_\Omega$ and $\operatorname{curl}(\exp(x^1), \exp(x^3), \exp(x^2)) \in X$, hence $X \neq 0$. Similarly, $\chi_\Omega \in X'$ and $\exp(x^1), \sin(x^2) \in X'$, so $X' \neq 0$. □

**Lemma 2** (Completeness)**.** $X, X'$ are Banach spaces.

*Proof.* Let $\{u_n\}$ be a Cauchy sequence in $X$. $\{u_n\}$ is a Cauchy sequence of $W^{m,1}_\sigma(\Omega) \cap W^{m,2}_\sigma(\Omega)$. Since $W^{m,1}(\Omega) \cap W^{m,2}(\Omega)$ are Banach spaces, $\{u_n\}$ converges in $W^{m,1}(\Omega) \cap W^{m,2}(\Omega)$. Let $u$ be the limit of $\{u_n\}$ in $W^{m,1}(\Omega) \cap W^{m,2}(\Omega)$. For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $l, n \geq N$, then $\|u_l - u_n\|_X < \varepsilon$. Using Fatou's lemma for counting measures,

$$\begin{aligned}
\|u - u_n\|_X &= \sum_{m=5}^{\infty} \sum_{k=0}^{\infty} \frac{1}{m!^4 (c'_{k,m})^2} \|u - u_n\|_{W^{m,1}(\Omega)\cap W^{m,2}(\Omega)} \\
&= \sum_{m=5}^{\infty} \sum_{k=0}^{\infty} \frac{1}{m!^4 (c'_{k,m})^2} \liminf_{l\to\infty} \|u_l - u_n\|_{W^{m,1}(\Omega)\cap W^{m,2}(\Omega)} \\
&\leq \liminf_{l\to\infty} \sum_{m=5}^{\infty} \sum_{k=0}^{\infty} \frac{1}{m!^4 (c'_{k,m})^2} \|u_l - u_n\|_{W^{m,1}(\Omega)\cap W^{m,2}(\Omega)} \\
&\leq \varepsilon.
\end{aligned}$$

Therefore,

$$\|u\|_X \le \|u - u_N\|_X + \|u_N\|_X \le \varepsilon + \|u_N\|_X < \infty,$$

hence $u \in X$. $\square$

**Lemma 3** (Separation of product)**.** There exists a constant $C_1 > 0$ such that

$$\left\|u^i v^i\right\|_{X'} \le C_1 \left\|u^i\right\|_{X'} \left\|v^i\right\|_{X'}$$

holds for all $u, v \in X'$.

*Proof.* For multi-indices $\alpha, \beta \in \mathbb{Z}^4_{\ge 0}$, let $c_{\alpha,\beta}$ denote the binomial coefficients. Define

$$c_\alpha = \sum_{\beta \le \alpha} c_{\alpha,\beta}.$$

There is a continuous embedding $X' \subset C^{k,\varepsilon}(\overline{\Omega})$ for any $k \in \mathbb{N}$ because

$$\begin{aligned}\|u_n - u\|_X \to 0 &\implies \|u_n - u\|_{W^{m,1}(\Omega)\cap W^{m,2}(\Omega)} \to 0\\ &\implies \|u_n - u\|_{C^{k,\varepsilon}(\overline{\Omega})} \to 0.\end{aligned}$$

([1]), so there exists a constant $c'_{k,m} > 0$ such that

$$\|u\|_{C^{k,\varepsilon}(\overline{\Omega})} \le c'_{k,m} \|u\|_{X'}.$$

By Leibniz formula,

$$\begin{aligned}\|\partial^\alpha (uv)\|_{L^p(\Omega)} &\le c_\alpha \|u\|_{C^{k,\varepsilon}(\overline{\Omega})} \|v\|_{C^{k,\varepsilon}(\overline{\Omega})} |\Omega|^{1/p}\\ &\le c_\alpha c'_{k,m} |\Omega|^{1/p} \|u\|_{X'} c'_{k,m} \|v\|_{X'}\\ &\le c_\alpha (c'_{k,m})^2 |\Omega|^{1/p} \|u\|_{X'} \|v\|_{X'},\end{aligned}$$

for $|\alpha| \le k$. Therefore, there exists a constant $C'_1 > 0$ such that

$$\|uv\|_{X'_{\text{new}}} \le C'_1 \|u\|_{X'_{\text{old}}} \|v\|_{X'_{\text{old}}}.$$

For any old $c'_{k,m}, c'_{k+1,m+1} \ge c'_{k,m} > 0$, there exists a new $c''_{k,m} := c'_{k,m}$. We correct : $c''_{k,m} \ge c'_{k,m} \ge \sqrt{\alpha_{0,5}} c''_{k,m}$.

We conjecture that one can prove Lemma 3 exactly for $c'_{k,m}$:
If there exists $0 < \alpha = \alpha_{c',c''} < 1$ such that

$$\alpha \|uv\|_{X'_{\text{old}}} \le \|uv\|_{X'_{\text{new}}} \le C'_1 \|u\|_{X'_{\text{old}}} \|v\|_{X'_{\text{old}}}.$$

Hence, it is sufficient to take $C_1 := C'_1/\alpha$ and

$$\|uv\|_{X'_{\text{old}}} \le C_1 \|u\|_{X'_{\text{old}}} \|v\|_{X'_{\text{old}}}.$$

To obtain $\alpha$, choose $0 < \alpha_{k,m} \leq 1$ so that

$$\alpha_{0,5} < \alpha_{k+1,m+1} \leq \alpha_{k,m},$$

and

$$0 < \frac{\alpha_{k,m}}{(c'_{k,m})^2} \leq \frac{1}{(c''_{k,m})^2}.$$

Using

$$\frac{\alpha}{(c'_{k,m})^2} \leq \frac{\alpha_{k,m}}{(c'_{k,m})^2} \leq \frac{1}{(c''_{k,m})^2}$$

and

$$\exp\left(1 + \frac{1}{\alpha_{1,6}}\right) \geq \exp\left(1 + \frac{1}{\alpha_{k,m}}\right) \geq \frac{1}{\alpha_{k,m}} \geq 1,$$

for

$$\alpha = \frac{1}{\inf_{k,m} \exp\left(1 + \frac{1}{\alpha_{k,m}}\right)}.$$

We expect that one can also choose $c'_{k,m}$ and justify the existence of $\alpha$ by using [10, Problem 9.6]: "If the linear space

$$X_{c',c''} := \left\{ u \in \bigcap_{m \geq 5} \bigcap_{p=1}^{2} W^{m,p}(\Omega) : \|u\|' < \infty, \|u\|'' < \infty \right\}$$

is complete with respect to the norms $\|\cdot\|'$ and $\|\cdot\|''$, and if $\|\cdot\|'' \leq \|\cdot\|'$, then these norms are equivalent."

□

**Lemma 4** (Absorption of differential)**.** There exists a constant $C_2 > 0$ such that

$$\|\partial_{x^j} u\|_X \leq C_2 \|u\|_X$$

for any $u \in S$.

**Lemma 5** (Boundedness of $S \ni u \mapsto E * (\chi_\Omega u) \in X$)**.** The map $S \ni u \mapsto E * (\chi_\Omega u) \in X$ is a bounded operator, so there exists a constant $C_3 > 0$ such that

$$\left\| \int_{\mathbb{R}\times\mathbb{R}^3} E(s,y)\chi_\Omega(t-s,x-y)u(t-s,x-y)\,ds\,dy \right\|_X \leq C_3\|u\|_X,$$

$$\left\| \int_{\mathbb{R}\times\mathbb{R}^3} E(s,y)\partial_{x^j}\chi_\Omega(t-s,x-y)u(t-s,x-y)\,ds\,dy \right\|_X \leq C_3 C_2\|u\|_X$$

hold for all $u \in S$.

*Proof.* As a function of $(s,y)$, for almost every $(t,x) \in \Omega$,

$$\begin{aligned}\operatorname{supp}(E(s,y)(\chi_\Omega u)(t-s,x-y)) &\subseteq -\overline{\Omega} + (t,x) \\ &= \big\{(s,y) \in \mathbb{R}\times\mathbb{R}^3 : (t-s,x-y) \in \Omega\big\}\end{aligned}$$

is the translation of the reverse of $\Omega$, so it is compact, and

$$\begin{aligned}\big|\partial_{t,x}^\alpha(E(s,y)(\chi_\Omega u)(t-s,x-y))\big| &\le E(s,y)\sup\big\{\big|\partial_{t,x}^\alpha u(t-s,x-y))\big| : (t,x) \in \Omega\big\} \\ &\le C_\alpha E(s,y) \in L^1_{s,y}(\Omega).\end{aligned}$$

Combining the theorem of differentiation under the integral sign, Hölder's inequality ([11]), and continuous embedding $S \subset L^\infty(\Omega)$, we have

$$\begin{aligned}\|\partial^\alpha(E*(\chi_\Omega u))\|_{L^p(\Omega)} &\le \|E*(\partial^\alpha(\chi_\Omega u))\|_{L^p(\Omega)} \\ &\le \Big\|\|E\|_{L^1(-\Omega+(t,x))}\|\partial^\alpha u(t-s,x-y)\|_{L^\infty(-\Omega+(t,x))}\Big\|_{L^p(\Omega)} \\ &\le \sup\Big\{\|E\|_{L^1(-\Omega+(t,x))} : (t,x)\in\Omega\Big\}|\Omega|^{1/p}\|\partial^\alpha u\|_{L^\infty(\Omega)} \\ &\le \sup\Big\{\|E\|_{L^1_{s,y}(-\Omega+(t,x))} : (t,x)\in\Omega\Big\}c'' C_2^{|\alpha|}|\Omega|^{1/p}\|u\|_X \\ &< \infty.\end{aligned}$$

Therefore, we obtain

$$\|E*(\chi_\Omega u)\|_X \le C_3\|u\|_X.$$

□

We take $C = \max\{C_1, C_2, C_3\}$. Note that Lemma 3, Lemma 4, and Lemma 5 hold for $C = O(|\Omega|)$. We take $\Omega, f, M > 0$ satisfying

$$C\|f\|_{X'} + 3C^3M^2 \le M.$$

We solve

$$\text{(N-S)'}\quad \partial_t u - \Delta u = f - (u\cdot\nabla)u,$$

that is, for any $a \in A$, there exist $u \in S$ and $\mathfrak{p} \in L^2(\Omega)$ such that

$$\begin{cases}\langle\partial_t u + (u\cdot\nabla)u - \Delta u + \nabla\mathfrak{p} - f, \varphi\rangle = 0 & (\varphi \in \mathcal{D}_\sigma(\Omega)), \\ \langle\operatorname{div}(u),\varphi\rangle = -\sum_{j=1}^3\big\langle u^j, \partial_{x^j}\varphi\big\rangle = 0 & (\varphi \in \mathcal{D}(\Omega)), \\ u(0,x) = a(x).\end{cases}$$

Define $\Phi\colon S \to X$ as follows:

$$\Phi[u](t,x) = \int_{\mathbb{R}\times\mathbb{R}^3} E(s,y)(\chi_\Omega P(f-(u\cdot\nabla)u))(t-s,x-y)\,ds\,dy.$$

We take a sequence $\{u_n\} \subset X$ as $u_0 \in S, u_1 \in S, u_1 - u_0 \in S$; if $n \geq 1$ then

$$\begin{aligned} u_{n+1}(t,x) &= \Phi[u_n](t,x) \\ &= \int_{\mathbb{R}\times\mathbb{R}^3} E(s,y)(\chi_\Omega P(f-(u_n\cdot\nabla)u_n))(t-s,x-y)\,ds\,dy. \end{aligned}$$

From Lemma 2, $X$ is a complete metric space. From Lemma 3, Lemma 4, Lemma 5, 5 and 6, the uniqueness and existence of a fixed point of $\Phi$ follow similarly to Banach's fixed point theorem ([5], [9]): There exists a unique $u \in X$ such that $\Phi[u] = u$.

**Lemma 6** (Well-definedness of $\Phi$)**.** For $u \in S$, we have

$$\|E * (\chi_\Omega P(f-(u\cdot\nabla)u))\|_X < \infty.$$

*Proof.* Since $\|P\| \leq 1$, we obtain

$$\begin{aligned} \|E * (\chi_\Omega P(f-(u\cdot\nabla)u))\|_X &\leq C\|f\|_{X'} + C\left\|u^1\partial_{x^1}u + u^2\partial_{x^2}u + u^3\partial_{x^3}u\right\|_{X'} \\ &\leq C\|f\|_{X'} + 3C^3M^2 \\ &< \infty. \end{aligned}$$

□

**Lemma 7** ($\Phi$ is a contraction mapping)**.** The map $\Phi\colon S \to X$ is Lipschitz continuous; that is, there exists a constant $L > 0$ such that, for all $u, v \in S$, if $u - v \in S$, then

$$\left\|\int_{\mathbb{R}\times\mathbb{R}^3} E(s,y)(\chi_\Omega P((v\cdot\nabla)v-(u\cdot\nabla)u)(t-s,x-y))\,ds\,dy\right\|_X \leq L\|u-v\|_X.$$

Moreover, the constant $L$ can be chosen so that $L < 1$.

*Proof.* Since

$$\begin{aligned} &((v\cdot\nabla)v-(u\cdot\nabla)u)(t-s,x-y) \\ &\qquad = \sum_{j=1}^{3}(v^j(\partial_{x^j}v - \partial_{x^j}u) + (v^j\partial_{x^j}u - u^j\partial_{x^j}u))(t-s,x-y), \end{aligned}$$

we have

$$\begin{aligned}
&\left\| \int_{\mathbb{R}\times\mathbb{R}^3} E(s,y)(\chi_\Omega P((v\cdot\nabla)v-(u\cdot\nabla)u))(t-s,x-y)\,ds\,dy \right\|_X \\
&\le C^2\|v\|_X \max_j \|\partial_{x^j}(v-u)\|_X + C^2\|v-u\|_X \max_j \|\partial_{x^j} u\|_X \\
&\le C^3 M\|v-u\|_X + C^3 M\|v-u\|_X \\
&\le 2C^3 M\|u-v\|_X.
\end{aligned}$$

Therefore, Lipschitz continuity follows for $L = 2C^3M$. □

**Lemma 8.** Let $U \in X$. Then $\langle U, \varphi\rangle = 0$ for all $\varphi \in \mathcal{D}_\sigma(\Omega)$ if and only if there exists a distribution $\mathfrak{p}$ such that $U = \nabla\mathfrak{p}$.

*Proof.* For any $\varphi \in \mathcal{D}_\sigma(\Omega)$, $\operatorname{div}(\varphi) = 0$, so by integration by parts

$$\begin{aligned}
\langle \nabla\mathfrak{p}, \varphi\rangle &= \int_\Omega \sum_{i=1}^3 (\nabla\mathfrak{p})^i(t,x)\varphi^i(t,x)\,dt\,dx \\
&= -\int_\Omega \mathfrak{p}(t,x)\operatorname{div}(\varphi)(t,x)\,dt\,dx \\
&= 0.
\end{aligned}$$

So, by Helmholtz decomposition, if $U = PU + \nabla\mathfrak{p}$, then $\langle U, \varphi\rangle = \langle \nabla\mathfrak{p}, \varphi\rangle = 0$. □

**Lemma 9** (Solvability of the Navier–Stokes equations)**.** The fixed point $u$ of $\Phi\colon S \to X$ is the solution of (N-S)'.

*Proof.* $u$ satisfies $\operatorname{div}(u) = 0$ in the sense of a distribution belonging to $\mathcal{D}'(\Omega)$. That is, for any $\varphi \in D(\Omega)$,

$$\langle \operatorname{div}(u), \varphi\rangle = -\sum_{j=1}^3 \langle u^j, \partial_{x^j}\varphi\rangle = 0.$$

In fact, for any $u \in W_\sigma^{m,p}(\Omega)$, there exists a Cauchy sequence $\{u_n\} \subset V_\sigma^{m,p}(\Omega)$. By integration by parts and Hölder's inequality, we have:

$$0 = -\sum_{j=1}^3 \langle u_n^j, \partial_{x^j}\varphi\rangle \to -\sum_{j=1}^3 \langle u^j, \partial_{x^j}\varphi\rangle.$$

Boundedness of $u$, $\partial_{x^j}u$ by the Sobolev embedding theorem and $|\Omega| < \infty$, we have $(u\cdot\nabla)u \in L^2(\Omega)$. So, by the Helmholtz decomposition, if we let $f = Pf + \nabla\mathfrak{f}$, $(u\cdot\nabla)u = P((u\cdot\nabla)u) + \nabla\mathfrak{u}$ then for any $\varphi \in \mathcal{D}_\sigma(\Omega)$, $\langle f, \varphi\rangle = \langle Pf, \varphi\rangle$, $\langle (u\cdot\nabla)u, \varphi\rangle = \langle P((u\cdot\nabla)u), \varphi\rangle$, hence we solve

$$\text{(N-S)'} \quad \partial_t u - \Delta u = f - (u\cdot\nabla)u \quad \text{in } \mathcal{D}'(\Omega).$$

By Proposition 1, the solution of the approximate equation on $\Omega$

$$\text{(N-S)''} \quad \partial_t v_n - \Delta v_n = P(f - (u_n \cdot \nabla)u_n)$$

satisfies

$$v_n = u_{n+1} = E * (\chi_\Omega P(f - (u_n \cdot \nabla)u_n)).$$

Therefore, the solution of (N-S)" satisfies

$$u_{n+1}(t,x) = \int_{\mathbb{R}\times\mathbb{R}^3} E(s,y)(\chi_\Omega P(f - (u_n \cdot \nabla)u_n))(t-s, x-y)\, ds\, dy.$$

$$\begin{aligned}(\partial_t - \Delta)u_{n+1}(t,x) &= \langle (\partial_t - \Delta)E(t-s, x-y), (\chi_\Omega(Pf - (u_n \cdot \nabla)u_n))(s,y)\rangle \\ &= \langle \delta(\tau) \otimes \delta(z), (\chi_\Omega P(f - (u_n \cdot \nabla)u_n))(t-\tau, x-z)\rangle \\ &= (P(f - (u_n \cdot \nabla)u_n))(t,x).\end{aligned}$$

By Hölder's inequality, $\|P\| \leq 1$, and the continuity of the product of $L^2(\Omega) \times L^2(\Omega) \ni (u,v) \mapsto uv \in L^1(\Omega)$ (see Remark 3), we have

$$\begin{aligned}&\left| \int_\Omega (P((u_n \cdot \nabla)u_n - (u \cdot \nabla)u))(t,x) \cdot \varphi(t,x)\, dt\, dx \right| \\ &\leq \|(u_n \cdot \nabla)u_n - (u \cdot \nabla)u\|_{L^1(\Omega)} \|\varphi\|_{L^\infty(\Omega)} \\ &\to 0 \quad (n \to \infty)\end{aligned}$$

By the continuity of the heat operator on $\mathcal{D}'(\Omega)$, we have

$$|\langle (\partial_t - \Delta)u_n, \varphi\rangle - \langle (\partial_t - \Delta)u, \varphi\rangle| \to 0 \quad (n \to \infty),$$

and hence $(\partial_t - \Delta)u = P(f - (u \cdot \nabla)u)$ holds. From Lemma 7, we have

$$u(t,x) = \int_{\mathbb{R}\times\mathbb{R}^3} E(s,y)(\chi_\Omega P(f - (u \cdot \nabla)u))(t-s, x-y))\, ds\, dy.$$

$u$ is a solution in the sense of the distribution in $\mathcal{D}'(\Omega)$ of (N-S)'. From Lemma 8, there exists $\mathfrak{p}$ such that $(\partial_t - \Delta)u + (u \cdot \nabla)u - f = -\nabla\mathfrak{p}$ holds. □

# 4 Properties of the solutions

**Lemma 10** (Vanishing)**.**

$$\lim_{t,|x|\to\infty} \partial^\alpha u(t,x) = 0.$$

*Proof.* $u$ is a measurable function on $\mathbb{R} \times \mathbb{R}^3$, so we can take the limits as $t, |x| \to \infty$.

$$\partial^\alpha u(t,x) = \int_\Omega E(t-s, x-y)(\partial^\alpha(P(f - (u \cdot \nabla)u)))(s,y)\, ds\, dy,$$

for any $t_0 > 0$, if $t - s > t_0$ then $|E(t-s, x-y)| \leq 1/t^{3/2}$, $\partial^\alpha(P(f - (u \cdot \nabla)u)) \in X \subset C^{0,\varepsilon}(\Omega)$ so $\lim_{t,|x|\to\infty} \partial^\alpha u(t,x) = 0$ follows from the bounded convergence theorem. □

The domain $\Omega$ can be taken arbitrarily large, so Lemma 10 can be thought of as a boundary condition.

**Lemma 11** (Continuity of $f \mapsto u$)**.** Let $f_n, f$ be such that $Pf_n, Pf \in S$ and $\|f_n - f\|_{X'} \to 0$. Let $u_n$ be the solutions corresponding to $f_n$ with initial value $a_n \in A$, and let be the solutions corresponding to $f$ with initial value $a \in A$. Then $\|u_n - u\|_X \to 0$.

*Proof.*

$$\begin{aligned}\|u_n - u\|_X = \Bigg\| &\int_{\mathbb{R}\times\mathbb{R}^3} E(s,y)(\chi_\Omega P(f_n - (u_n \cdot \nabla)u_n))(t-s, x-y))\, ds\, dy \\ &- \int_{\mathbb{R}\times\mathbb{R}^3} E(s,y)(\chi_\Omega P(f - (u_n \cdot \nabla)u))(t-s, x-y))\, ds\, dy \Bigg\|_X \\ &\leq C\|f_n - f\|_X + 2C^3 M \|u_n - u\|_X.\end{aligned}$$

So

$$\limsup_{n\to\infty} \|u_n - u\|_X \leq 2C^3 M \limsup_{n\to\infty} \|u_n - u\|_X.$$

$$\limsup_{n\to\infty} \|u_n - u\|_X \leq 2M.$$

Therefore

$$0 \leq (1 - 2C^3 M) \limsup_{n\to\infty} \|u_n - u\|_X \leq 0.$$

Hence

$$\limsup_{n\to\infty} \|u_n - u\|_X = \lim_{n\to\infty} \|u_n - u\|_X = 0.$$

□

**Proposition 3** (Elementary weak solutions as elements of Bochner class)**.** Let $I = \{t \in \mathbb{R} : \exists x \in \mathbb{R}^3, (t,x) \in \Omega\}$, $\Omega' = \{x \in \mathbb{R}^3 : \exists t \in \mathbb{R}, (t,x) \in \Omega\}$. For $1 \leq p < \infty$,

$$a \in C^\infty(\Omega') \cap L^p_\sigma(\Omega'), u \in C(I; L^p_\sigma(\Omega')).$$

*Proof.* Since $u \in C^{0,\varepsilon}(\Omega)$, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $(t, x), (t', x) \in \Omega$, if $|(t, x) - (t', x)| < \delta$ then $|u(t, x) - u(t', x)| < \varepsilon$. Therefore

$$\left\| u(t, \cdot) - u(t', \cdot) \right\|_{L^p(\Omega')} \leq \left| \Omega' \right|^{1/p} \varepsilon.$$

□

**Lemma 12** (Elementary Weak Solutions are non-zero on the boundary of the domain)**.** An elementary weak solution satisfies

$$u(t, x) = \int_{\mathbb{R} \times \mathbb{R}^3} E(s, y)(\chi_\Omega P(f - (u \cdot \nabla)u))(t - s, x - y))\, ds\, dy.$$

For each $(t, x)$, define a function $U^{(t,x)}$ of the variables $(s, y)$ by

$$U^{(t,x)}(s, y) = (\chi_\Omega P(f - (u \cdot \nabla)u))(t - s, x - y).$$

If $\operatorname{supp} U^{(t,x)} \subset \overline{\Omega}$ if and only if $\operatorname{supp} u \subset \overline{\Omega}$, then $u$ satisfy the energy inequality. (by integration by parts) Since $\Omega \subset \mathbb{R} \times \mathbb{R}^3$ is bounded, there exists $R > 0$ such that

$$(t, x) \in \Omega \implies |t|, |x| < R$$

In particular, if $|s| > R$ or $|y| > R$ then $(s, y) \notin \Omega$. Let $(t_0, x_0) \in \operatorname{supp} U^{(t,x)} \cap \overline{\Omega}$. Then there exists $R' > 0$ such that

$$|t_0 - s| + |x_0 - y| < R' \implies (s, y) \in \operatorname{supp} U^{(t,x)}.$$

On the other hand, we have

$$\begin{aligned} |t_0 - s| + |x_0 - y| &\geq ||t_0| - |s|| + ||x_0| - |y|| \\ &= ||s| - |t_0|| + ||y| - |x_0||. \end{aligned}$$

Hence, if $|s|$ and $|y|$ are sufficiently large (for instance, $|s| > R$ and $|y| > R$), then

$$|t_0 - s| + |x_0 - y| > R',$$

so, $(s, y) \notin \operatorname{supp} U^{(t,x)}$, while $(s, y) \notin \Omega$. This shows that $\operatorname{supp} U^{(t,x)} \subset \overline{\Omega}$ does not hold.

## 5 A Fixed Point Theorem

**Proposition 4.** Let $S$ be a closed subset of a Banach space such that $S \neq \{0\}$ and $u - v \in S$ for all $u, v \in S$. Let $\Phi \colon S \to S$ be a mapping for which there exists a constant $0 < L < 1$ such that

$$\|\Phi[u] - \Phi[v]\| \leq L \|u - v\|$$

for all $u, v \in S$. Then there exists a unique fixed-point $u \in S$ of $\Phi$.

*Proof.* Define a sequence $\{u_n\} \subset S$ by choosing $u_0 \in S$ (for instance, $u_0 = 0$) and setting $u_{n+1} = \Phi[u_n]$. By the assumption of $\Phi$, the sequence $\{u_n\}$ is a Cauchy in $S$. Since $S$ is closed in a Banach space, it follows that $\{u_n\}$ converges to some $u \in S$. This limit $u$ is a fixed-point of $\Phi$.

Suppose that $u$ and $v$ are fixed points of $\Phi$. Then

$$\|u - v\| = \|\Phi[u] - \Phi[v]\| \leq L\|u - v\|.$$

Iterating this inequality, we obtain

$$\|u - v\| \leq L^n\|u - v\|.$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$, it follows that $\|u - v\| = 0$, and hence $u = v$. □

# 6 Remarks

**Remark 1.** We can solve semi-linear partial differential equations with constant coefficients

$$Lu = f(u)$$

on $S' = \{u \in X' : \|u\|_{X'} \leq M\}$ similarly if $L$ has locally integrable fundamental solutions. For example, the non-linear heat equation, the non-linear Schrödinger equation, the non-linear wave equation with 3-dimension, and the KdV equation.

**Remark 2.** As functions $\varphi$ that $\operatorname{div}(\varphi) = 0$, it is sufficient to take any $\psi \in \mathcal{D}(\Omega)$ and set $\varphi = \operatorname{curl}\psi$.

**Remark 3.** Let $\|u_n - u\|_{L^2(\Omega)} \to 0$, $\|v_n - v\|_{L^2(\Omega)} \to 0$. By the triangle inequality, we have

$$\left|\|u_n\|_{L^2(\Omega)} - \|u\|_{L^2(\Omega)}\right| \leq \|u_n - u\|_{L^2(\Omega)}$$

for any sufficiently large $n$. On the other hand, $\|u_n\|_{L^2(\Omega)} < \|u\|_{L^2(\Omega)} + 1$. Therefore

$$\begin{aligned}\|u_n v_n - uv\|_{L^1(\Omega)} &\leq \|u_n\|_{L^2(\Omega)}\|v_n - v\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}\|u_n - u\|_{L^2(\Omega)} \\ &< (\|u\|_{L^2(\Omega)} + 1)\|v_n - v\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}\|u_n - u\|_{L^2(\Omega)} \\ &\to 0 \quad (n \to \infty).\end{aligned}$$

**Remark 4.** We change the assumptions: $0 \in I$, $0 \notin \Omega'$. Let $\rho(0, \cdot) \in X'(\Omega')$. We can solve

$$\begin{cases}\partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \operatorname{Div}(D(u) + \operatorname{div}(u)(\delta^{ij}) - \mathfrak{p}(\delta^{ij})) = f - \rho(\partial_t u + (u \cdot \nabla)u)\end{cases}$$

similarly. Here,

$$(\operatorname{Div} T)^i = \sum_{j=1}^{3} \partial_{x^j} T^{ij}, D^{ij}(u) = \partial_{x^j} u^i + \partial_{x^i} u^j.$$

The operator $L_u\colon S' \ni \rho \mapsto \operatorname{div}(\rho u) \in X'$ is bounded, so

$$\rho = e^{-tL_u}\rho(0, \cdot).$$

Then

$$\operatorname{Div}(D(u) + \operatorname{div}(u)(\delta^{ij}) - \mathfrak{p}(\delta^{ij})) = f - e^{-tL_u}\rho(0, \cdot)(\partial_t u + (u \cdot \nabla)u).$$

The fundamental solutions of the elliptic operators are locally integrable on new $\Omega$. Using the above lemmas, the unique existence, smoothness, and properties of the solutions follow similarly.

## 7 Supplement

This is an application of functional analysis to the existence and smoothness of the Navier–Stokes equations using elementary weak solutions in Sobolev spaces.

We solve the problem in mathematics. The problems are not in physics, so we do not use any physics or assumptions-falsified mathematics, such as in other papers. We use mathematics only. We can solve the problem by using an exactly and completely FALSIFIED resolution, where large initial values destroy the earth, because uniqueness does NOT hold, or SMALL initial values love your cup of coffee.

There are no long or complicated calculations; semi-groups, a priori estimates, and boundary conditions on $\Omega' \subset \mathbb{R}^3$ are not used at all. We apply the local solvability of linear partial differential operators with constant coefficients ([8], [3]), erase the pressure p by Helmholtz projection $P$, and solve

$$\frac{\partial u}{\partial t} - \Delta u = Pf - P\left((u_n \cdot \nabla)u_n\right),$$
$$\frac{\partial u}{\partial t} - \Delta u = Pf - P\left((u \cdot \nabla)u\right),$$

in $X = \bigcap_{m \geq 5,\, p=1,2} W^{m,p}(\Omega)$, $\Omega$ is an open set in the universal space $\mathbb{R} \times \mathbb{R}^3$ that has a smooth boundary.

For example, cylinder $\Omega = I \times \Omega' = (-T, T) \times B_3(0, R)$ includes the earth. We handle the variables $t, x$ simultaneously. Cylinders are Lipschitz domains, so we can use the Sobolev embedding theorem on $\Omega$ ([1]) and the Helmholtz decomposition on $\Omega'$ ([6]).

The initial value or external force term $f$ is small as norms $\|f\|_X < \frac{1}{12C^4}$, but we can take $\Omega$ as (the earth) $\Subset \Omega$.

The non-zero examples of elements of $X$ as a normed space do not depend on the choice of $c'_{k,m}$.

For any old $c'_{k,m}, c'_{k+1,m+1} \geq c'_{k,m} > 0$,

As a result, we can take $c'$ as a positive constant: take $c''$ as $5000c'$, take $\alpha_{0,5}$ as $10^{-23}$.

We can construct many non-zero solutions by choosing $c'$.

There exists $C = O(|\Omega|) > 0$, for example,

$$\begin{aligned}
\|(x_i \mapsto \sin x_i)(x_j \mapsto \cos x_j)\|_X &\leq C\|\sin\|_X\|\cos\|_X, \\
\left\|(x_i \mapsto e^{\frac{1}{2}x_i})(x_j \mapsto e^{\frac{1}{3}x_j})\right\|_X &\leq C\left\|x_i \mapsto e^{\frac{1}{2}x_i}\right\|_X\left\|x_j \mapsto e^{\frac{1}{3}x_j}\right\|_X, \\
\left\|\partial_j^k e^x\right\|_X &\leq C^k\|e^x\|_X, \\
\left\|\partial_j^k \sin x\right\|_X &\leq C^k\|\sin x\|_X.
\end{aligned}$$

for $i, j = 1, 2, 3$.

$$\|\partial_j u_n\|_X \leq CM$$

even if $n \geq 2$, and

$$\|\partial_j(u_{n+1} - u_n)\|_X \leq C\|u_{n+1} - u_n\|_X,$$

for $n \geq 1$.

So, there exists $0 < L < 1$, by mathematical induction, for $n \geq 2$,

$$\|u_{n+1} - u_n\|_X \leq L^{n-1}\|u_2 - u_1\|_X,$$

even if $u_n - u_{n-1} \notin S$ and $u_1 \neq \Phi[u_0]$.

Our map $\Phi$ is well-defined: the completion Sobolev space and the common Sobolev space are equivalent to each other, and the convolution of the functions does not depend on anything outside $\Omega$.

(To construct the solutions,

$$\|\partial_j(u_{n+1} - u_n)\|_X \leq C\|u_{n+1} - u_n\|_X, \quad j = 1, 2, 3$$

It is not necessary, but we can easily calculate its use. The proof is similar to the construction of a unique solution under this description.

Therefore, existence follows similarly to the proof of Banach's fixed point theorem on a closed set $S \subset X_\sigma$. The solutions $u$, p satisfy

$$\frac{\partial u}{\partial t} - \Delta u = Pf - P\left((u \cdot \nabla)u\right)$$

in the sense that distributions in $X_\sigma$ are

$$\mathrm{div}(u) = 0$$

in the sense that distribution is in $X$.

$f \neq 0 \implies u \neq 0$ and $f = 0, a = 0 \implies u = 0$. There exists a non-zero solution by Fujita, Kato ([4]) or Shibata ([6]), even if $f = 0$, so elementary weak solutions are different from the solutions by Fujita, Kato or Shibata.

$u|_{\partial\Omega'} \neq 0$, but by Helmholtz decomposition, if $\Omega = I \times \Omega' = (-T, T) \times B_3(0, R)$, then $(u \cdot n)|_{\partial\Omega'} = 0$, so our solutions satisfy the energy inequality on $\Omega = (-T, T) \times B_3(0, R)$. By integration by parts, we obtain

$$(\partial_t u - \Delta u) \cdot u = (f - \nabla \mathrm{p} - (u \cdot \nabla)u) \cdot u.$$

Hence

$$\frac{1}{2}\partial_t(u \cdot u) - (\Delta u) \cdot u = f \cdot u - ((u \cdot \nabla)u) \cdot u.$$

Integrating over $B = B_3(0, R)$ and $t \in (0, t)$, we obtain

$$\begin{aligned}\frac{1}{2}\|u(t)\|^2_{L^2(B)} - \int_0^t \int_{\partial\Omega'} |u \cdot n|\,(1 + |\nabla u|^2)\, dA\, ds + \int_0^t \|\nabla u(s)\|^2_{L^2(B)}\, ds \\ \leq \frac{1}{2}\|u(0)\|^2_{L^2(B)} + \int_0^t (f(s), u(s))_{L^2(B)}\, ds.\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\|u(t)\|^2_{L^2(B)} + \int_0^t \|\nabla u(s)\|^2_{L^2(B)}\, ds \\ \leq \frac{1}{2}\|u(0)\|^2_{L^2(B)} + \int_0^t (f(s), u(s))_{L^2(B)}\, ds.\end{aligned}$$

Our solutions are smooth, unique, $u$ is defined on $\mathbb{R} \times \mathbb{R}^3$, and

$$\lim_{t, |x| \to \infty} \partial^\alpha u(t, x) = 0,$$

for any multi-index $\alpha$. The map $f \mapsto u$ is continuous. The initial value satisfies

$$a \in C^\infty(\Omega') \cap L^p(\Omega').$$

The solution satisfies

$$u \in C(I; L^p(\Omega')).$$

If

$$\int_0^T (f(s), \varphi(s))_{L^2(\Omega')}\, ds + (a, \varphi(0))_{L^2(\Omega')} = 0$$

for all test functions $\varphi$, and

$$u|_{\partial\Omega'} = 0,$$

then

$$f = 0, \quad a = 0, \quad u = 0.$$

satisfy the energy inequality and $u$ belongs to the Serrin class. So $u$ is the unique Leray-Hopf's weak solution and an elementary weak solution.

Therefore, if $u$ is the Leray-Hopf's weak solution and an elementary weak solution, then $u = 0$.

If we change the definition of $S$ by

$$S = \left\{ u \in X : \|u\|_X \leq M, \quad \left\|\partial_j^k u\right\|_X \leq C^k \|u\|_X, \quad j = 1, 2, 3, k \geq 0 \right\},$$

then the existence follows exactly.

$$\partial_j : X \to W^{m,p}(\Omega)$$

is a closed operator, so $S$ is a closed set by the proof using Fatou's lemma and

$$\left| \|\partial_j u_n\|_X - \|\partial_j u\|_X \right| \leq \|\partial_j u_n - \partial_j u\|_X \to 0.$$

(similar to the proof of Lemma 2)

Uniqueness follows. For $f \neq 0$, we can construct many non-zero solutions for which uniqueness holds by Baire's category theorem: construct $u \in S$ as an interior point in $X_\sigma$, construct the open neighborhood $B_u$, and think $v \in S \cap B_u$ satisfies $\|u - v\| < M$, and prove similarly to "Uniqueness" for $u - v$ by using $C_n$, $C_n \to \infty$, $u_{n(k)} \to u$, and $v_{n(k)} \to v$ from

$$\begin{aligned} \left\|u - u_{n(k)}\right\|, \left\|v - v_{n(k)}\right\| &< \frac{1}{k}, \\ C_{n(k)}\left\|u_{n(k)} - v_{n(k)}\right\| &\leq \left\|\partial_j (u_{n(k)} - v_{n(k)})\right\| \\ &= \left\|\partial_j u_{n(k)} - \partial_j v_{n(k)}\right\| \\ &\leq \left\|\partial_j u_{n(k)}\right\| + \left\|\partial_j v_{n(k)}\right\| \\ &\leq 2CM. \end{aligned}$$

We conjecture that blow-up solutions can also be constructed. For any $\varepsilon > 0$, there exist $\delta > 0$, a constant $M(\varepsilon) > 0$, and a function $f \in S_{M(\varepsilon)}$ such that:

- $M(\varepsilon_1) > M(\varepsilon_2)$ whenever $\varepsilon_1 > \varepsilon_2$,
- $CM(\varepsilon) > \|E * \chi_\Omega P f\| > 2\varepsilon$,
- $|\Omega| < \delta \implies \left\|E * \chi_\Omega P\big((u \cdot \nabla)u\big)\right\| < \varepsilon$.

By the triangle inequality, $\|u\| = \|\Phi[u]\| > \varepsilon$. For example,

$$\begin{aligned} f(t,x) &= 2(\varepsilon + 1)(\|E * \chi_\Omega P \sin(x_1)\| + 1)\sin(x_1), \\ M(\varepsilon) &= 2 + \exp(2(\varepsilon + 1)(\|E * \chi_\Omega P \sin(x_1)\| + 1)). \end{aligned}$$

We conjecture that we can take the initial value a large by making $|\Omega|$ smaller.

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Parts of this work were developed through discussions and materials shared on x.com (account: @Ohrui_PDE_LowD), as well as on mathlog.info and note.com, and through activities at the Tokyo University of Science. We are grateful to the many individuals who provided feedback and support for this research during the period from February 5, 2022, to December 30, 2025. In particular, this manuscript has been read by Naofumi Mori (@n_mori00), Associate Professor of Mathematical Sciences, as well as by two Japanese users of X, @R1sFs and @ksvzvsk.

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# Justification

Masatoshi Ohrui

May 24, 2026

## 1 Lemma 2

$\sum_{k\geq 0, m\geq 5} 1/(c''^2_{k,m} m!^4) = C' < \infty$,

$1/(c'^2_{k,m}) \leq (1/(c''^2_{k,m}))(c''^2_{k,m} m!^4)/(c'^2_{k,m})$.

Therefore, $(1/(C'+1))\|u_m - u_n\|' \leq \|u_m - u_n\|'' < \varepsilon$.

Thus $\exists C' > 0, \|u_m - u_n\|' \leq (C'+1)\varepsilon$.

## 2 Lemma 4

If $\forall\alpha, \partial^\alpha_{t,x} U \neq 0$, i.e. , if $U$ is not polynomials function as $t, x$, simultaneously, then by mean value theorem on integration, we can correct the values of the functions outside of $\Omega$ that for any $\varphi \in \mathcal{D}(\Omega)$,

$\int_\Omega \partial^\alpha (E * \chi_\Omega U)\varphi$

$= (-1)^{|\alpha|} \int_\Omega (E * \chi_\Omega U)\, \partial^\alpha \varphi$

$= (-1)^{|\alpha|} \int_\Omega \int_{-\Omega+(t,x)} E(s,y)U(t-s, x-y) ds dy\, \partial^\alpha \varphi(t,x) dt dx$

$= (-1)^{|\alpha|} \int_\Omega \int_\Omega E(s,y)U(t-s, x-y) ds dy\, \partial^\alpha \varphi(t,x) dt dx$

$= \int_\Omega \int_\Omega E(s,y) \partial^\alpha U(t-s, x-y) ds dy\, \varphi(t,x) dt dx.$

# Supplement and References Written in Japanese Language

Masatoshi Ohrui

June 5, 2026

## 1 Japanese books

To justify of Lemma 3, take $c'_{k,m}$ as $c'_{k,m} \geq 1$.

Hisashi Okamoto, Principal of the Navier-Stokes Equations, University of Tokyo Press, 2023, ISBN: 978-4-13-061315-6;

Yoshihiro Shibata, Basic of Mathematical Fluids 1, 2, Iwanami books, 2022, ISBN: 9784000298582, 9784000298599 .

# supplement

June 9, 2026

## 1 About other PDE and embedding constants

We can apply our Elementary PDE Theory to the wave equations with dimension 2, by Theorem 10.2.1., Lars Hörmander, "The Analysis of Linear Partial Differential Operators II", Springer, 1983.

$\|u\|_{C^{0,\varepsilon}(\overline{\Omega})} \leq \|u\|_{C^{k,\varepsilon}(\overline{\Omega})} \leq c''_{k,m}\|u\|_X$ and $\|u\|_{C^{0,\varepsilon}(\overline{\Omega})} \leq c''_{0,5}\|u\|_X$.

Therefore we can take $c''_{k,m} \leq c_{0,5}$ as a constant. (Reference: Haim Brezis, "Functional Analysis, Sobolev Spaces and Partial Differential Euations", Springer, 2010)