Proof of Hartog's Phenomenon and Cohomology Vanishing Theorem

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May 12, 2026

Abstract

We can prove Hartog's phenomenon by solving the $\bar{\partial}$-equation for compactly supported forms. To solve the equation, we construct the solution using convolution.

Proposition 1 (Existence of solutions of $\bar{\partial}$-equation)**.** Let $f \in C^{\infty}_{0,(p,q+1)}(\mathbb{C}^n)$ satisfy

$$f = \sum_{I,J} f_{I,J}\, dz^I \wedge d\bar{z}^J,$$

$f_{I,J} \in C_0^{\infty}(\mathbb{C}^n)$ for $n \geq 2$ and $\bar{\partial} f = 0$.

Then there exist (p,q)-form u such that

$$\bar{\partial} u = f,$$

$u_{I,J} \in C^{\infty}_{0,(p,q)}(\mathbb{C}^n)$.

Proof. The identity

$$\frac{\partial}{\partial \bar{z}}\left(\frac{1}{\pi z}\right) = \delta$$

holds in the sense of distributions ([2]).

Let

$$f_1 = \sum_{I,J} f_{I,J}\, dz^I \wedge d\bar{z}^{J\setminus\{j_1\}},$$

and define

$$u_1(z) = \iint \frac{f_{I,J}(z_{j_1} - w)}{\pi w}\, dw \wedge d\bar{w}.$$

Then

$$\frac{\partial u_1}{\partial \bar{z}_{j_1}} = f_{I,J}.$$

Hence

$$v_1(z) = \sum_{I,J} \left(\iint \frac{f_{I,J}(z_{j_1} - w)}{\pi w}\, dw \wedge d\bar{w} \right) dz^I \wedge d\bar{z}^{J\setminus\{j_1\}}$$

satisfies

$$\bar{\partial} v_1 = \pm f.$$

□

Proposition 2 (Hartogs' phenomenon)**.** Let Ω be an open subset of $\mathbb{C}^n$. Let $K \subset \Omega$ be compact, and assume that $\Omega \setminus K$ is connected. Then, for every $u \in A(\Omega \setminus K)$, there exists $\mathfrak{u} \in A(\Omega)$ such that

$$\mathfrak{u} = u \qquad \text{on } \Omega \setminus K.$$

The proof of Proposition 2 is similar to that of [1, Theorem 2.3.2]. In the proof, the function v plays the role of the solution, and the open set may be taken to be the interior of $\Omega \setminus K$. Note that Proposition 2 does not hold when $n = 1$.

Using

$$(\bar{\partial} u)|_\Omega = f|_\Omega,$$

we can prove the vanishing theorem for $\bar{\partial}$-cohomology on an n-dimensional complex manifold Ω.

Acknowledgement

This paper has been checked by 2 persons.

References

[1] Lars Hörmander, *An Introduction to Complex Analysis in Several Variables*, third ed., North Holland, 1973.

[2] Elias M. Stein and Rami Shakarchi, *Functional Analysis: Introduction to Further Topics in Analysis*, Princeton University Press, 2011.