

On Gravitational Binding Energy Evaluation

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The aim of this work is to determine the gravitational energy that would be released if a galaxy's stellar content were to coalesce in an idealized, perfectly inelastic collision. This energy can be interpreted as the total gravitational energy stored within the galaxy. Under general relativity, this stored energy contributes to the curvature of spacetime and could thus explain the gravitational effect generally attributed to dark matter.

The corresponding binding energy equation is derived in two independent ways. The first derivation is purely Newtonian and remains valid across nearly forty orders of magnitude in mass. The second employs the Newtonian gravitational work generated by accretion onto a black hole. Despite their distinct derivations, both lead to the same mathematical structure, which can be extended to galactic systems. When applied to galaxies, this equation establishes a direct link between their dynamical properties and their stellar composition.

This link offers a way to test the equation. We modify its algebraic form and examine the behavior of the dynamical projection—derived from SPARC—within the stellar population space. We then compare this projection with GALEX and SDSS photometry. The preferred forms yield clear, persistent photometric correlations that are robust enough to recover stellar population relationships comparable in quality to those of established empirical models. Furthermore, this correspondence breaks down when the natural association between baryonic and dark mass is severed.

Taken together, these results make gravitational binding energy a viable candidate for explaining the origin of dark mass. We then examine the cosmological implications of this interpretation and propose a mechanism for the exchange between gravitational energy and vacuum energy to reconcile theoretical tensions with the different conditions that prevailed in earlier cosmic epochs.

1 Introduction

The origin of the dark mass inferred from galactic dynamics remains one of the central open questions in astrophysics [8]. Its presence is robustly established through multiple independent observations, including rotation curves and dynamical modeling [23, 24], as well as gravitational lensing measurements [1, 7]. However, its physical nature remains unknown.

At the same time, gravitational theory, both Newtonian and relativistic, has demonstrated remarkable accuracy across a wide range of scales. This raises the possibility that the observed discrepancy can be accounted for without invoking exotic matter or modified gravity, but by recognizing that gravitational potential energy is not straightforward to define and evaluate as an intrinsic contribution to the effective mass budget of astrophysical systems.

In standard formulations, gravitational potential energy is defined relative to an arbitrary reference level, typically taken at infinity. As a result, the expression $E = -GmM/d$ is well suited for describing reversible orbital dynamics, but it does not directly provide an intrinsic energy associated with the formation or internal organization of an extended many-body system. More generally, the notion of potential energy becomes ambiguous when applied to complex systems. Related ambiguities have long been recognized in classical mass renormalization [3] and in geometrical definitions of mass in general relativity [14, 16, 21].

A useful analogy is provided by nuclear binding energy. The mass of a helium nucleus is smaller than the combined mass of the hydrogen nuclei from which it may be formed, and this mass deficit is entirely accounted for by the energy released during the fusion process. The binding energy can therefore be determined by comparing two configurations containing the same conserved constituents, one configuration initially dispersed and the other tightly bound.

The primary objective of this work is to apply the same accounting principle to the gravitational organization of a galaxy. We calculate the gravitational energy that would be released if all its stars underwent a hypothetical perfectly inelastic aggregation into a single compact body, while accounting exclusively for gravitational energy. We investigate whether the mass equivalent of this released energy can entirely account for the dark mass inferred from galactic dynamics, just as the energy released during nuclear fusion accounts for the mass deficit of the resulting nucleus.

We define gravitational binding energy as the energy released when a distributed configuration is transformed into a maximally compact reference configuration while conserving total mass and constituent volumes. The corresponding solid sphere may be viewed as a configuration in which all constituents are placed side by side without altering their internal structure. It therefore introduces no additional compression of the constituents and provides a well-defined reference state for evaluating the gravitational energy released by their hypothetical perfectly inelastic aggregation.

The theoretical development proceeds in two complementary directions. We first derive the binding-energy equation entirely within Newtonian gravity and examine

the conditions under which this construction can be extended from ordinary gravitational systems to the galactic scale. The limits of this model are then considered explicitly, together with the continuation of the same energy accounting when aggregation leads to black-hole formation. We then obtain the same functional form through an independent construction compatible with general relativity. The two derivations start from different physical descriptions but converge toward the same equation. This agreement provides a key argument for the theory’s consistency before it is confronted with galactic observations.

The theoretical predictions are then confronted to a sequence of observational tests. The first concerns the relationship between baryonic and dark mass. The binding-energy equation predicts a dependence proportional to $M^{5/3}$. When the exponent is allowed to vary so that the galaxies are projected within the theoretically admissible stellar-population bounds, the preferred values converge toward $5/3$, in agreement with the predicted exponent.

A second test concerns the stellar-population component of the equation. The relative contributions of its different terms are varied independently, and the resulting projections are compared with observed GALEX and SDSS colors. The best photometric agreement is obtained for coefficients close to the theoretical predictions, indicating that the theoretical mathematical structure is necessary to produce a good empirical fit.

The next test examines whether the projection depends on the actual galaxy-by-galaxy association between baryonic and dark mass. These pairings are randomly permuted across the sample while preserving the global mass distributions. The population-space correspondence then collapses, showing that the observed relation between M_{bar} and M_{dark} carries essential information for the projection.

Stellar populations inferred solely from dynamics calculated using SPARC data are then directly compared with photometric data from GALEX and SDSS; this approach yields models of quality comparable to those that are already well-established. Furthermore, using photometric data to refine the projection enables the generation of unambiguous correlations. This test alone is sufficient to refute the null hypothesis; the energy equation indeed allows for the construction of a projection of dynamics into the space of stellar populations. Demonstrating the existence of this relationship—previously thought not to exist—is the central argument validating our work. When all tests are combined, even loosely, the null hypothesis must be rejected.

Finally, we examine the cosmological consequences of this interpretation. Extending the gravitational-energy calculation to earlier cosmic epochs reveals potential tensions associated with the evolution of stellar populations and gravitationally bound structures. We show that these difficulties can be addressed through a simple extension in which gravitational and vacuum energy are allowed to exchange while their combined energy–momentum remains conserved. This provides a possible connection between the binding energy, the cosmological dark sector, and a global thermodynamic description of the Universe.

1.1 Useful gravitational potential energy

If we consider the Newtonian gravitational force equation, $F = GmM/d^2$, integration from d to infinity gives the usual potential-energy expression $E_p = -GmM/d$. This expression is well suited to describing orbital dynamics, but its absolute value depends on the arbitrary choice of the zero of gravitational potential. For the purpose considered here, the physically relevant quantity is therefore the energy difference between two specified configurations of the same system.

For example, moving a mass m from a distance d to $d + h$ from a mass M gives:

$$\Delta E_p = E_p(d + h) - E_p(d) = \frac{GmMh}{d^2 + dh} \quad (1)$$

This energy difference is invariant under an additive transformation of the Newtonian potential and is therefore independent of the arbitrary choice of potential zero.

We now generalize this construction to two classical macroscopic bodies. The Newtonian formulation is considered over a broad domain extending approximately from the microgram scale, 10^{-9} kg, to the mass of the Sun, $M_\odot \simeq 2 \times 10^{30}$ kg. This interval spans nearly forty orders of magnitude. The lower scale is chosen conservatively so that the objects can be treated unambiguously as macroscopic classical bodies, without requiring a quantum description. The upper scale remains safely within the weak-field domain for ordinary material densities, where the equivalent compact configuration remains far outside its Schwarzschild radius. This provides a broad classical domain in which the construction can be established directly from Newtonian gravity before considering its continuation into the relativistic regime.

Consider two bodies of masses m and M , radii r and R , and volumes:

$$v = \frac{4\pi r^3}{3} \quad V = \frac{4\pi R^3}{3}$$

We define a compact reference configuration by conserving both total mass and total constituent volume:

$$M_t = m + M \quad V_t = V + v = \frac{4\pi}{3}(R^3 + r^3)$$

The corresponding equivalent radius is:

$$R_t = (R^3 + r^3)^{1/3}$$

If the two bodies have the same volumetric density, the compact body has the same density:

$$\frac{m}{v} = \frac{M}{V} = \frac{M_t}{V_t}$$

If their densities differ, conservation of total mass and total volume uniquely determines the effective density of the equivalent compact sphere. Within this Newtonian weak-field domain, the gravitational potential energy of a homogeneous solid sphere is given by:

$$E_p = -\frac{3GM^2}{5R} \quad (2)$$

The gravitational binding-energy scale can then be evaluated by comparing the compact and separated configurations, their energies are denoted directly by:

$$E_{\text{compact}} = -\frac{3GM_t^2}{5R_t} \quad (3)$$

$$E_{\text{separated}} = -\frac{3GM^2}{5R} - \frac{3Gm^2}{5r} - \frac{GMm}{d} \quad (4)$$

The corresponding energy difference is:

$$\Delta E_p = E_{\text{separated}} - E_{\text{compact}} \quad (5)$$

$$= \left(-\frac{3GM^2}{5R} - \frac{3Gm^2}{5r} - \frac{GMm}{d} \right) + \frac{3GM_t^2}{5R_t} \quad (6)$$

With this convention, ΔE_p measures the energy needed to take the system from the compact reference state to the separated state. The reverse process gives the same quantity a more direct physical meaning. If the two separated bodies are allowed to fall together under their mutual gravity and merge through a perfectly inelastic collision, an amount of energy equal to ΔE_p is released. As they approach one another, gravitational potential energy is converted into kinetic energy. In the construction considered here, however, the volumes and internal structures of the constituents are preserved. The released energy therefore cannot be stored in the final body through compression, deformation, or internal heating and must leave the system as radiation. In this sense, this modeling provides a concrete interpretation of ΔE_p as the gravitational energy liberated by the hypothetical aggregation of the two bodies.

In this framework, the conservation of mass and volume allows for the definition of a unique compact reference state, without implying the disappearance of microscopic interactions within a real aggregate. The comparison is deliberately limited to the gravitational component by holding the total mass and constituent volume fixed; this makes it possible to exclude non-gravitational contributions from the energy balance.

The preceding derivation establishes the binding energy across nearly forty orders of magnitude. That interpretation must eventually fail as the mass and compactness increase, since the radius of the compact configuration approaches the Schwarzschild scale. What changes at that point is the justification of the expression rather than necessarily the expression itself. As developed later, an independent construction compatible with general relativity leads to the same functional form in the regime where the compact reference state can no longer be described by Newtonian gravity alone.

1.2 Mass-equivalent gravitational binding energy

The gravitational binding energy introduced above is a genuine energy difference between two configurations of the same physical system. During aggregation, this energy may be released and therefore contributes directly to the system’s energy balance, just as any other form of energy would. Its gravitational origin does not alter this status and does not prevent the released energy from being associated with an equivalent mass through $E = mc^2$.

In general relativity, the gravitational field of a system depends on its total mass–energy content rather than on the rest masses of its constituents alone. A change in that energy content therefore modifies the gravitational contribution of the system. If an amount of energy ΔE_p is released from or added to the system, the associated change in gravitating mass can be written through mass–energy equivalence as:

$$M_{B_E} = \frac{\Delta E_p}{c^2} \quad (7)$$

This representation does not require any additional material component. It simply expresses the contribution of gravitational binding energy in mass units. For an isolated system, a change in total energy changes the mass parameter that determines its external gravitational field by the corresponding amount E/c^2 . At the level of the gravitational mass inferred from observations, this energy contribution is therefore indistinguishable from an equivalent amount of mass–energy producing the same spacetime curvature.

Gravitational energy nevertheless occupies a special place in general relativity. Unlike the energy carried by matter, it cannot generally be assigned a unique local energy density that is independent of the coordinate system or observer. Its contribution must instead be characterized through global or quasi-local measures of the mass–energy of the gravitating system. This distinction appears explicitly in relativistic constructions such as the Hawking mass, the Misner–Sharp mass, and related definitions of quasi-local mass [14, 16, 21].

A useful point of comparison is the gravitational self-interaction framework developed by Deur [9, 10]. In this approach, the nonlinear self-coupling of the gravitational field in General Relativity strengthens the binding of massive systems and can account for part of the excess gravitational response usually attributed to dark matter. This additional contribution can then be represented as an effective dark mass, without requiring a new material component.

This suggests that the self-interaction contribution calculated by Deur and the gravitational potential-energy contribution considered here may be two descriptions of the same underlying gravitational energy. Such an identification is physically plausible, but to our knowledge it has not been established explicitly. Deur’s treatment derives the change in the gravitational field and the resulting dynamics from gravitational self-interaction, whereas the present framework approaches the same question through the energy associated with gravitational binding.

One important difference is that the self-interaction calculations do not directly yield a simple closed-form expression for this energy contribution in terms of the masses and volumes of the constituents. The formulation developed here provides such an estimator directly from the gravitational binding-energy difference. Demonstrating a quantitative equivalence between the two approaches would therefore establish a concrete link between them and help determine whether they are describing the same gravitational-energy contribution.

The quantity M_{BE} provides a direct mass representation of the gravitational binding energy introduced in the preceding section. Expressed in this form, the contribution can be compared directly with the baryonic mass and with the additional gravitating mass inferred from galactic dynamics, without treating it as a separate material component.

The existence of a potential-energy difference does not depend on whether the corresponding transformation occurs spontaneously. A stable system may remain indefinitely in one configuration while retaining a real energy difference relative to another accessible state. Nuclear fission offers a familiar example. Uranium nuclei in a stable mineral do not need to fission spontaneously for the associated energy difference to exist. If fission is induced, that energy can be released and measured. The same reasoning applies to nuclear fusion and, more simply, to an object resting on a table. Its stability does not remove its gravitational potential energy, which is released if the object is allowed to fall.

The same reasoning applies to a stationary or collisionless gravitational system. The absence of ongoing aggregation does not remove the energy difference between its distributed and compact configurations. If the bodies are instead allowed to fall together and merge, that difference becomes physically manifest as released gravitational energy. The binding-energy difference therefore exists independently of whether the system has undergone, is undergoing, or will ever undergo the corresponding transformation.

1.3 Newtonian examples of gravitational binding energy

To give a sense of the magnitude of the gravitational binding energy defined above, we consider three familiar systems that remain well within the Newtonian regime, namely the Moon–Earth, Earth–Sun, and Jupiter–Sun systems. For each case, the observed separated state is compared with its hypothetical volume-conserving compact counterpart, allowing the corresponding gravitational energy difference to be evaluated directly.

The two bodies are assigned masses m and M , radii r and R , and a separation d , with m denoting the smaller mass. The compact configuration has total mass $M_t = m + M$ and radius $R_t = (R^3 + r^3)^{1/3}$, as required by conservation of the combined volume. The physical parameters used for the three systems are listed in Table 1.

For each system, the energy difference is defined as $\Delta E_p = E_{\text{separated}} - E_{\text{compact}}$, with the corresponding mass equivalent given by $M_{BE} = \Delta E_p/c^2$. The calculated

Table 1: Physical parameters used for the three Newtonian binding-energy examples. The compact reference radius R_t is obtained by conserving the combined volume of the two bodies.

System	m (kg)	M (kg)	r (m)	R (m)	d (m)	R_t (m)
Moon–Earth	7.35×10^{22}	5.97×10^{24}	1.74×10^6	6.38×10^6	3.8×10^8	6.42×10^6
Earth–Sun	5.97×10^{24}	1.99×10^{30}	6.38×10^6	6.96×10^8	1.5×10^{11}	6.96×10^8
Jupiter–Sun	1.90×10^{27}	1.99×10^{30}	6.99×10^7	6.96×10^8	7.8×10^{11}	6.97×10^8

values are reported in Table 2. The last column expresses this mass equivalent as a fraction of the smaller body’s mass m .

Table 2: Gravitational energy differences between the observed separated systems and their hypothetical volume-conserving compact configurations. The corresponding mass equivalent is $M_{BE} = \Delta E_p/c^2$, while M_{BE}/m gives this energy scale in mass units relative to the smaller body.

System	E_{compact} (J)	$E_{\text{separated}}$ (J)	ΔE_p (J)	M_{BE} (kg)	M_{BE}/m
Moon–Earth	-2.2795×10^{32}	-2.2413×10^{32}	3.8176×10^{30}	4.2477×10^{13}	5.8×10^{-10}
Earth–Sun	-2.2751×10^{41}	-2.2751×10^{41}	1.3024×10^{36}	1.4491×10^{19}	2.4×10^{-6}
Jupiter–Sun	-2.2787×10^{41}	-2.2751×10^{41}	3.5519×10^{38}	3.9520×10^{21}	2.1×10^{-6}

The scale of these energies becomes more intuitive when compared with familiar physical benchmarks. For the Moon–Earth system, the hypothetical aggregation releases approximately 3.82×10^{30} J. Taking a large thermonuclear explosion as a reference, this corresponds to roughly 40 trillion such events. Through mass–energy equivalence, the same amount of gravitational energy has a mass equivalent of approximately 4.25×10^{13} kg.

For the Earth–Sun system, the corresponding energy reaches approximately 1.30×10^{36} J, with a mass equivalent of 1.45×10^{19} kg. For Jupiter–Sun, these values increase to 3.55×10^{38} J and 3.95×10^{21} kg, respectively. Even in these familiar Newtonian systems, the absolute gravitational binding energy is therefore substantial, although its mass equivalent remains only a small fraction of the material mass involved.

The Earth–Sun result is especially noteworthy because a mass-equivalent contribution of order 10^{19} kg is substantial on terrestrial gravitational scales. Whether such a contribution could produce a measurable effect near the Earth would depend on how the corresponding gravitational energy is distributed throughout the Earth–Sun system. The global scalar value alone is therefore insufficient to determine the strength of any local gravitational signature.

The gravitational binding energy is a property of the interacting system as a whole and cannot be assigned uniquely to either component. It can nevertheless be normalized by either mass to characterize the gravitational energy scale associated with each body. For the interaction term $U = -GmM/d$, division by the smaller mass

gives $|U|/m = GM/d$, while division by the larger mass gives $|U|/M = Gm/d$. These two specific energy scales therefore differ by the mass ratio M/m .

When $M \gg m$, the same interaction corresponds to a much larger energy scale when expressed relative to the smaller body's mass. This does not mean that the gravitational energy is localized within that body. It remains a property of the full interacting system, while the normalization simply reflects the fact that the smaller body lies in a gravitational environment dominated by the larger mass.

The ratio M_{BE}/m in Table 2 should be understood in the same way. It expresses the mass-equivalent binding energy of the entire system relative to the smaller body's mass, without implying that this energy belongs specifically to that body.

1.4 Quasi-localization of gravitational binding energy

An important question is how the gravitational binding energy of a system should be associated spatially. For a system of n bodies, this energy arises from the interactions among bodies belonging to the same bound configuration. It is therefore more naturally associated with a closed surface enclosing the system as a whole than with any one of its individual components.

General relativity gives this idea a more precise form through the concept of quasi-local energy. Although gravitational energy cannot generally be assigned a unique covariant density at each point, several relativistic constructions associate the total mass–energy contained within a finite region with its enclosing two-surface [15, 25]. The connection is especially transparent in spherical symmetry. The Misner–Sharp energy measures the mass–energy enclosed by a spherical surface and remains unchanged as that surface is extended through a surrounding Schwarzschild vacuum. This agrees with Birkhoff's theorem, under which the exterior geometry of a spherically symmetric isolated system depends only on the mass enclosed within the region.

The spherical construction introduced below does not require the galaxy itself to be considered spherical. In a disk galaxy, matter located beyond a given cylindrical radius may still contribute to the radial gravitational field. The quantity of interest can nevertheless be represented by the spherically equivalent mass inferred from the circular velocity:

$$M_{\text{eq}}(R) = \frac{RV^2(R)}{G} \quad (8)$$

This is the spherical mass that would produce the same radial acceleration at R . Since the baryonic rotation velocity already accounts for the actual spatial distribution of the baryons, it follows that:

$$M_{\text{bar,eq}}(R) = \frac{RV_{\text{bar}}^2(R)}{G} \quad (9)$$

provides an equivalent radial description without assuming a spherical baryonic distribution.

The same representation can be used for the effective mass associated with gravitational binding energy. At each radius, the radial dynamics can therefore be described in terms of the equivalent baryonic source together with the effective binding-energy mass associated with that quasi-local region. Binding energy belonging to a larger configuration remains associated with the larger region that encloses it and is not reassigned to smaller radii.

2 Newtonian gravitational binding energy

The first point to note is that the interaction term $-GmM/d$ in ΔE_p quickly becomes negligible compared with the internal self-energy terms when the system contains well-separated gravitational scales. Relative to the dominant contribution GM^2/R , its magnitude is $(GMm/d)/(GM^2/R) = (m/M)(R/d)$. For astrophysical bodies such as planets and stars, both $m/M \ll 1$ and $R/d \ll 1$, leaving the interaction term only a very small contribution.

This separation of scales persists as the construction is carried through successive hierarchical levels. Bound objects occupy only a small fraction of the volume set by their mutual separations. Conserving the combined constituent volume gives an effective radius that grows as $N^{1/3}$, while the characteristic spacing between objects follows approximately the same scaling. The ratio between occupied size and separation therefore remains small from one level of aggregation to the next. In a Milky Way-type galaxy, for example, the total stellar volume corresponds to an effective radius much smaller than typical interstellar separations, with $R/d \sim 10^{-4}$.

At galactic masses, the volume-conserving compact configuration may eventually reach a regime where a Newtonian physical interpretation is no longer appropriate. This does not affect the scale-separation argument itself. At this stage, the effective radius is used only as a geometrical measure of the total constituent volume, and no Newtonian binding energy is assigned to the hypothetical compact galactic state.

The effective binding-energy contribution considered here is not obtained by treating the conventional pairwise term $-GmM/d$ as a positive mass contribution. In the hierarchical regime, that interaction term becomes subdominant, while the dominant contribution arises from the compact-reference self-energy terms proportional to M^2/R . To leading order, the two-body energy difference can therefore be written as:

$$\Delta E_p = \frac{3G}{5} \left(\frac{M_t^2}{R_t} - \alpha_M \frac{M^2}{R} - \alpha_m \frac{m^2}{r} \right) \quad (10)$$

where α_M and α_m describe the internal mass distributions of the two bodies. For homogeneous spheres $\alpha = 1$, while more generally $\alpha \sim \mathcal{O}(1)$ depends on the density profile.

Consider a system of n bodies with masses m_i , radii r_i , and internal coefficients α_i , combined successively under volume conservation. After the i th aggregation, the

accumulated mass is $M_i = \sum_{j \leq i} m_j$, and the corresponding effective radius satisfies $R_i^3 = R_{i-1}^3 + r_i^3$. The energy difference introduced at each step can then be written as:

$$\Delta E_i = \frac{3G}{5} \left(\frac{M_i^2}{R_i} - \frac{M_{i-1}^2}{R_{i-1}} - \alpha_i \frac{m_i^2}{r_i} \right), \quad \Delta E_p = \sum_{i=1}^n \Delta E_i \quad (11)$$

Because the sum telescopes, the intermediate compact-state contributions cancel, leaving only the final collective term together with the initial internal contributions:

$$\Delta E_p = \frac{3G}{5} \left[\frac{M_n^2}{R_n} - \sum_{i=1}^n \alpha_i \frac{m_i^2}{r_i} \right] \quad (12)$$

with $M_n = \sum_i m_i$ and $R_n = (\sum_i r_i^3)^{1/3}$. The first term depends only on the total mass and conserved volume of the system, while the second retains the internal binding contributions of the individual bodies. In sufficiently large systems, these individual terms become asymptotically negligible and, once the intrinsic binding energies of the constituents are included, cancel exactly.

The compact, volume-conserving configuration is the most compact state that still preserves the internal volumes and structures of the constituents. It may be pictured as a state in which the bodies have been brought into contact without undergoing compression. This constraint is essential because any further contraction would introduce additional physical effects, including degeneracy pressure, radiation pressure, and material stresses, into the energy balance. By stopping the construction at volume conservation, the comparison remains restricted to the gravitational energy difference between the separated and aggregated configurations.

The construction is therefore designed to avoid introducing an action–reaction contribution into the energy balance.

2.1 Proof of permutation invariance

The final result must also be independent of the order in which the bodies are aggregated. Let σ be any permutation of $\{1, \dots, n\}$. Applying Equation (12) to this ordering gives:

$$\Delta E_p(\sigma) = \frac{3G}{5} \left[\frac{(\sum_{i=1}^n m_{\sigma(i)})^2}{(\sum_{i=1}^n r_{\sigma(i)}^3)^{1/3}} - \sum_{i=1}^n \alpha_{\sigma(i)} \frac{m_{\sigma(i)}^2}{r_{\sigma(i)}} \right] \quad (13)$$

Each sum is invariant under permutation, so $\Delta E_p(\sigma) = \Delta E_p$. The binding-energy difference is therefore independent of the aggregation order. $\blacksquare$

2.2 Simplified calculation with identical bodies

Consider n identical bodies, each with mass m , radius r , and internal structure characterized by a coefficient α . After the i th aggregation, the accumulated mass is $M_i = im$. Since volume is conserved and the bodies have identical density, the corresponding effective radius is $R_i = i^{1/3}r$.

The energy released at each aggregation step is:

$$\Delta E_i = \frac{3G}{5} \left(\frac{M_i^2}{R_i} - \frac{M_{i-1}^2}{R_{i-1}} - \alpha \frac{m^2}{r} \right), \quad \Delta E_p = \sum_{i=1}^n \Delta E_i \quad (14)$$

The intermediate terms cancel telescopically, leaving:

$$\Delta E_p = \frac{3G}{5} \left(\frac{M_n^2}{R_n} - \alpha n \frac{m^2}{r} \right) = \frac{3G}{5} \left(n^{5/3} - \alpha n \right) \frac{m^2}{r} \quad (15)$$

where $M_n = nm$ and $R_n = n^{1/3}r$. The collective contribution therefore scales as $n^{5/3}$, whereas the sum of the internal binding terms grows only as n . Their relative contribution is $\alpha n^{-2/3}$ and vanishes as n becomes large.

The assembly energy is thus increasingly dominated by the collective term M_n^2/R_n . In the large- n limit, the gravitational binding energy depends primarily on the total mass and effective radius of the system, with the internal structure of the individual constituents contributing only a diminishing correction.

The result becomes even simpler when the intrinsic binding energies already carried by the initial bodies are included. The quantity ΔE_p accounts for the energy released by assembling the n bodies, while each constituent begins with its own gravitational binding energy. Their combined intrinsic contribution is $(3G/5)\alpha nm^2/r$. Restoring this contribution to the total energy balance gives:

$$E_{\text{tot}} = \Delta E_p + \frac{3G}{5} \alpha n \frac{m^2}{r} = \frac{3G}{5} \frac{M_n^2}{R_n} \quad (16)$$

The internal-structure contribution is removed in two distinct ways. For large n , its relative importance decreases as $n^{-2/3}$ compared with the collective term. Once the intrinsic binding energies already present in the initial bodies are included, however, the same contribution cancels exactly from the full energy balance.

The resulting expression is therefore independent of α and takes the same form as the gravitational binding energy of a homogeneous self-gravitating sphere with total mass M_n and volume-conserving radius R_n . Only the global mass and volume of the aggregated system remain, with no dependence on the detailed internal structure of its constituents.

2.3 Generalization to arbitrary distributions

The same result extends naturally to non-identical bodies. Consider n constituents with masses m_i , radii r_i , and internal-structure coefficients α_i . Under volume conservation, the cumulative mass and effective radius after the i th aggregation are $M_i = \sum_{j=1}^i m_j$ and $R_i = (\sum_{j=1}^i r_j^3)^{1/3}$. Applying the telescoping sum derived above then gives:

$$\Delta E_p = \frac{3G}{5} \left[\frac{(\sum_{i=1}^n m_i)^2}{(\sum_{i=1}^n r_i^3)^{1/3}} - \sum_{i=1}^n \alpha_i \frac{m_i^2}{r_i} \right] \quad (17)$$

This expression is entirely deterministic and does not depend on any particular distribution of masses, radii, or internal structures. The constituents may therefore be drawn from normal, log-normal, truncated, empirical, or other distributions, provided the relevant moments exist. Equation (17) also places no requirement on the statistical independence of mass and radius.

If the constituents are instead viewed as random variables drawn from a statistically homogeneous population, their large- n behavior follows from $\sum_i m_i \simeq n\langle m \rangle$ and $\sum_i r_i^3 \simeq n\langle r^3 \rangle$. The collective contribution then scales as $n^{5/3}\langle m \rangle^2/\langle r^3 \rangle^{1/3}$, whereas the internal term grows only as $n\langle \alpha m^2/r \rangle$. Their relative contribution therefore scales as:

$$\mathbb{E}[\Delta E_p] \simeq \frac{3G}{5} \left[n^{5/3} \frac{\langle m \rangle^2}{\langle r^3 \rangle^{1/3}} - n \left\langle \alpha \frac{m^2}{r} \right\rangle \right] \quad (18)$$

The ratio of the internal to the collective contribution therefore scales as $n^{-2/3}$, with coefficient $\langle \alpha m^2/r \rangle \langle r^3 \rangle^{1/3} / \langle m \rangle^2$. Provided the relevant moments remain finite and $\alpha \sim \mathcal{O}(1)$, this ratio vanishes as $n \rightarrow \infty$. The asymptotic suppression obtained for identical bodies therefore extends to arbitrary component distributions.

The deterministic result also remains valid for multimodal populations. Different modes may be treated as distinct subpopulations, either combined into a single weighted population or evaluated separately and then summed.

The exact cancellation remains unchanged. Once the intrinsic binding energies of all initial constituents are restored, the total energy becomes:

$$E_{\text{tot}} = \Delta E_p + \frac{3G}{5} \sum_{i=1}^n \alpha_i \frac{m_i^2}{r_i} = \frac{3G}{5} \frac{(\sum_{i=1}^n m_i)^2}{(\sum_{i=1}^n r_i^3)^{1/3}} \quad (19)$$

The same double protection therefore extends to arbitrary distributions. In the assembly energy, the internal-structure contribution is suppressed as $n^{-2/3}$, while in the complete binding-energy balance it cancels exactly. The final result depends only on the total mass and the volume-conserving effective radius of the aggregated system, not on the detailed internal structure of its constituents.

2.4 Derivation of the full binding-energy formula

These results can now be extended to hierarchical systems in which bodies are first grouped into homogeneous classes and then combined at a larger scale. For each homogeneous class of bodies i , let M_i , V_i , and R_i denote its total mass, total constituent volume, and corresponding effective radius. If v_{ik} is the volume of the k th body in that class, then $V_i = \sum_{k=1}^{N_i} v_{ik} \simeq N_i \langle v \rangle_i$. The corresponding effective density is therefore $\rho_i = M_i/V_i \simeq \langle m \rangle_i / \langle v \rangle_i$, which gives:

$$R_i = \left(\frac{3M_i}{4\pi\rho_i} \right)^{1/3} \quad (20)$$

Equation (19) can then be applied separately to each homogeneous class of bodies. Summing the resulting binding-energy contributions over all classes gives the total class-level term:

$$E_{\text{sum}} = \frac{3G}{5} \sum_{i=1}^n \frac{M_i^2}{R_i} \quad (21)$$

These class-level binding energies remain part of the system when the classes of bodies are combined, so the next hierarchical level contributes an additional term rather than replacing the previous ones.

In principle, this new contribution could be obtained by evaluating the $(n^2 - n)/2$ gravitational relations among the class-level subsystems. When the number of subsystems is large and no preferred ordering is introduced, however, the continuum limit recovers the same functional form as a compact homogeneous configuration with total mass $M = \sum_i M_i$ and volume-conserving radius $R = (\sum_i R_i^3)^{1/3}$. The resulting global aggregation term is therefore $(3G/5)M^2/R$.

The complete Newtonian hierarchical binding energy can then be written as:

$$E_p = \frac{3G}{5} \left[\sum_{i=1}^n \frac{M_i^2}{R_i} + \frac{M^2}{R} \right] \quad (22)$$

The first term represents the binding energy already contained within the homogeneous class-level subsystems, while the second describes the additional binding generated when those subsystems are combined at the scale of the complete system.

Each class-level aggregation also contains formal intrinsic terms proportional to $\sum_j m_{ij}^2/r_{ij}$. As established in Sections 2.2 and 2.3, these terms become negligible as $N^{-2/3}$ for large populations and cancel exactly when the intrinsic binding energies of the individual bodies are restored. They therefore do not appear as separate terms in Equation (22).

The global contribution does not depend on the number of classes used to represent the system. The classes provide only a finite description of its constituent bodies, whereas M^2/R captures the continuum-limit aggregation of the underlying gravitational subsystems.

2.5 Relativistic limit of the Newtonian construction

Equation (22) can be carried naturally from one level of structure to the next. Consider a system S assembled from subsystems S_i that are already gravitationally bound. The energy associated with those lower-level structures is not lost when they become part of a larger system. It remains embedded in the total energy balance, while the new level contributes the additional binding produced by assembling the subsystems themselves. Repeating the construction therefore builds the total binding energy progressively, with each hierarchical level adding its own contribution to those already present.

This reasoning also shows where the Newtonian description must eventually break down. For a compact configuration of mass M and radius R , the binding-energy scale $B_E = (3G/5)M^2/R$ corresponds through mass–energy equivalence to $M_{B_E} = B_E/c^2$. Introducing the Schwarzschild radius $R_s = 2GM/c^2$ gives:

$$\frac{M_{B_E}}{M} = \frac{3}{10} \frac{R_s}{R} \quad (23)$$

As R approaches R_s , gravitational self-energy can no longer be treated as a small correction to the total mass–energy budget, and the weak-field Newtonian approximation progressively loses its validity. For a static, spherically symmetric, and isotropic fluid configuration, the Buchdahl bound $R > 9R_s/8$ imposes an even more restrictive condition:

$$\frac{M_{B_E}}{M} < \frac{4}{15} \quad (24)$$

The compact, volume-conserving configuration must therefore cease to represent a Newtonian material state before its binding energy can increase without limit.

The recursive construction can also be examined in the limiting case of successive black-hole absorption. Before crossing the horizon, each bound subsystem carries the mass–energy associated with its internal structure. After absorption, that structure and the history of its assembly are no longer separately accessible in the final black hole. Their energetic contribution, however, cannot simply disappear, since energy conservation requires it to remain included in the mass–energy of the final object. The disappearance of observable substructure therefore does not erase the contributions inherited from earlier levels of aggregation.

This limiting picture allows the recursive structure established in the Newtonian regime to be followed into successive relativistic aggregation. In the construction below, each stellar class is assigned a gravitational length scale defined by an equivalent mean density, which is then used to evaluate the energy carried through successive absorption. The scale follows only from the equality of mean densities; it does not represent a material boundary or a physical tidal-disruption radius. The construction is therefore not a fully general-relativistic derivation. Despite this, it leads independently to the same functional form as the Newtonian calculation.

3 Binding-energy derivation in the relativistic regime

Consider a black hole that successively absorbs stars from a population class i characterized by a common intrinsic mean density ρ_i . At a given stage, let $\mathcal{M}$ be the mass already incorporated into the black hole, and let the next star have mass $m_\star$ and radius $r_\star$, such that $\rho_i = 3m_\star/(4\pi r_\star^3)$.

For each incoming star, we define a density-equivalent tidal scale $r_{t,i}$ by requiring the mean density formally associated with the accumulated black-hole mass $\mathcal{M}$ at that scale to equal the intrinsic mean density of the stellar class:

$$\frac{\mathcal{M}}{r_{t,i}^3} = \frac{m_\star}{r_\star^3}$$

Using $\rho_i = 3m_\star/(4\pi r_\star^3)$ gives:

$$r_{t,i}(\mathcal{M}) = \left(\frac{3\mathcal{M}}{4\pi\rho_i} \right)^{1/3}$$

The density-equivalent tidal scale therefore depends on the accumulated black-hole mass and on the intrinsic density of the stellar class, but not on the mass of the individual star. It is a gravitational length scale defined solely through the density equivalence above, with no requirement that it coincide with a physical tidal-disruption surface or with any material boundary of the absorbing system.

The corresponding gravitational-energy increment associated with this scale is:

$$dE_{p,i} = \frac{G\mathcal{M}}{r_{t,i}(\mathcal{M})} d\mathcal{M} = G \left(\frac{4\pi\rho_i}{3} \right)^{1/3} \mathcal{M}^{2/3} d\mathcal{M} \quad (25)$$

For a stellar class containing a sufficiently large number of constituents, the sequence of individual absorption events can be replaced by its continuum limit. Integrating over the accumulated mass from zero to the total class mass M_i , and introducing the intrinsic volume-equivalent radius $R_i = (3M_i/4\pi\rho_i)^{1/3}$, gives:

$$\begin{aligned} E_{p,i} &= G \left(\frac{4\pi\rho_i}{3} \right)^{1/3} \int_0^{M_i} \mathcal{M}^{2/3} d\mathcal{M} \\ &= \frac{3G}{5} \frac{M_i^2}{R_i} \end{aligned} \quad (26)$$

The M_i^2/R_i dependence is therefore recovered without requiring the stellar population to occupy a compact material sphere. The factor $3/5$ emerges from the cumulative gravitational energy evaluated at the density-equivalent scale introduced above. The result follows solely from the relation between accumulated mass and the intrinsic density of the stellar class, without assigning $r_{t,i}$ the meaning of a physical disruption radius.

If the same density-equivalent scale is related instead to a detailed tidal-disruption model, the internal structure of the stars, their orbital properties, and the precise disruption criterion may introduce an order-unity correction factor ξ_i , such that:

$$r_{t,i} = \xi_i \left(\frac{3\mathcal{M}}{4\pi\rho_i} \right)^{1/3}$$

The same calculation then gives:

$$E_{p,i} = \xi_i^{-1} \frac{3G}{5} \frac{M_i^2}{R_i}$$

The density-equivalence condition $\mathcal{M}/r_{t,i}^3 = m_\star/r_\star^3$ corresponds exactly to $\xi_i = 1$; it is therefore fixed by the definition itself rather than introduced as an independently fitted normalization. Order-unity modifications of the equivalent scale may change the coefficient, but leave the functional dependence $E_{p,i} \propto M_i^{5/3} \rho_i^{1/3} = M_i^2/R_i$ unchanged.

The accumulated energy is also independent of the order in which the constituents are absorbed. For a finite sequence, the contribution associated with the a th absorption is proportional to $\mathcal{M}_a^{5/3} - \mathcal{M}_{a-1}^{5/3}$, where $\mathcal{M}_a$ is the accumulated mass after that event. Summing over the full sequence produces a telescoping series that depends only on the final class mass M_i . The continuum integral represents the same order-independent accumulation.

Summing the contributions from all stellar population classes then gives:

$$E_{p,\text{class}} = \frac{3G}{5} \sum_{i=1}^n \frac{M_i^2}{R_i} \quad (27)$$

3.1 Derivation of the global binding term

The class-level derivation applies separately to populations whose constituents share a common intrinsic density. At the next hierarchical level, stellar classes with different intrinsic densities are combined, so no single class-specific density-equivalent scale is sufficient to describe the system. We therefore define the total mass as $M = \sum_i M_i$ and the intrinsic volume-equivalent radius through $R^3 = \sum_i R_i^3$. This quantity is neither the physical radius of the galaxy nor a disk scale length. It is defined solely from the conserved intrinsic volumes of the stellar constituents.

Because the global binding contribution is a state function, it may be evaluated along any convenient self-similar assembly path. Let $M_i(\lambda) = \lambda M_i$ with $0 \leq \lambda \leq 1$, while keeping the intrinsic densities fixed. It then follows that $M(\lambda) = \lambda M$ and $R(\lambda) = \lambda^{1/3} R$. The resulting collective gravitational work is:

$$E_{p,\text{global}} = \int_0^1 \frac{GM(\lambda)}{R(\lambda)} dM(\lambda) = \frac{GM^2}{R} \int_0^1 \lambda^{2/3} d\lambda = \frac{3G}{5} \frac{M^2}{R} \quad (28)$$

The factor $3/5$ therefore follows from the same mass–volume scaling that appears in the class-level calculation. No homogeneous spherical configuration of the galaxy is required. The radius R serves only as the intrinsic volume scale associated with the collective hierarchical degree of freedom.

Combining the independently obtained class-level and global contributions gives:

$$E_p = \frac{3G}{5} \left[\sum_{i=1}^n \frac{M_i^2}{R_i} + \frac{M^2}{R} \right] \quad (29)$$

This construction can remain applicable when the final absorbing system enters the relativistic regime because it never requires a Newtonian material configuration whose effective radius falls below its own Schwarzschild radius. In the class-level derivation, the density-equivalent scale is instead a gravitational reference scale fixed by the intrinsic stellar density and the accumulated mass. It is not required to represent either a material surface or an externally accessible tidal-disruption radius. The absorption process itself is relativistic, whereas the density-equivalent construction sets the gravitational-energy scaling carried into that process. The same functional form thus emerges as in the independent Newtonian hierarchical derivation.

4 Mass-fraction form of the binding-energy equation

The binding-energy expression contains two distinct contributions, one associated with the stellar classes and the other with their global aggregation. To keep these two roles explicit in the analytical developments that follow, we write:

$$E_p = \frac{3G}{5} \left[\alpha \sum_{i=1}^n \frac{M_i^2}{R_i} + \beta \frac{M^2}{R} \right] \quad (30)$$

Here α and β weight the class-level and global contributions, respectively. The physical derivation developed above corresponds to $\alpha = \beta = 1$.

Let $\phi_i = M_i/M$ be the mass fraction of stellar class i , so that $M_i = \phi_i M$ and $\sum_i \phi_i = 1$. Substituting $R_i = (3M_i/4\pi\rho_i)^{1/3}$ into the class-level term gives:

$$\sum_{i=1}^n \frac{M_i^2}{R_i} = \left(\frac{4\pi}{3} \right)^{1/3} M^{5/3} \sum_{i=1}^n \rho_i^{1/3} \phi_i^{5/3} \quad (31)$$

For the global contribution, volume conservation gives $R^3 = \sum_i R_i^3$. Using $R_i^3 = 3M_i/(4\pi\rho_i)$ and $M_i = \phi_i M$ gives $R^3 = (3M/4\pi) \sum_i \phi_i/\rho_i$, and consequently:

$$\frac{M^2}{R} = \left(\frac{4\pi}{3} \right)^{1/3} M^{5/3} \left(\sum_{i=1}^n \frac{\phi_i}{\rho_i} \right)^{-1/3} \quad (32)$$

Substituting both contributions into Equation (30) gives:

$$E_p(M, \phi; \alpha, \beta) = \frac{3G}{5} \left(\frac{4\pi}{3} \right)^{1/3} M^{5/3} \left[\alpha \sum_{i=1}^n \rho_i^{1/3} \phi_i^{5/3} + \beta \left(\sum_{i=1}^n \frac{\phi_i}{\rho_i} \right)^{-1/3} \right] \quad (33)$$

For any constant values of α and β , the dependence on the total mass remains exactly factorized as $M^{5/3}$. Dividing the binding energy by this mass scale therefore removes the dependence on the total baryonic mass:

$$\frac{E_p}{M^{5/3}} = \frac{3G}{5} \left(\frac{4\pi}{3} \right)^{1/3} \left[\alpha \sum_{i=1}^n \rho_i^{1/3} \phi_i^{5/3} + \beta \left(\sum_{i=1}^n \frac{\phi_i}{\rho_i} \right)^{-1/3} \right] \quad (34)$$

The normalized binding energy is therefore determined entirely by the stellar-population composition through the fractions ϕ_i and intrinsic densities ρ_i . The total baryonic mass enters only through the overall $M^{5/3}$ scaling. Equivalently, writing the mass-equivalent contribution as $M_{BE} = E_p/c^2$ leaves $M_{BE}/M^{5/3}$ dependent only on the stellar population. Galaxies with the same stellar-population composition thus share the same normalized binding-energy contribution, irrespective of their total mass.

4.1 Loose analytical bounds

The mass-fraction formulation also yields bounds that are independent of the detailed stellar-population composition. Let $\rho_{\min} = \min_i \rho_i$ and $\rho_{\max} = \max_i \rho_i$, with $\phi_i \geq 0$ and $\sum_i \phi_i = 1$. For $\alpha \geq 0$ and $\beta \geq 0$, the two population-dependent terms are bounded by $\sum_i \phi_i^{5/3} \in [n^{-2/3}, 1]$ and $\sum_i \phi_i/\rho_i \in [1/\rho_{\max}, 1/\rho_{\min}]$. The normalized binding energy must therefore satisfy:

$$\rho_{\min}^{1/3} n^{-2/3} \leq \sum_{i=1}^n \rho_i^{1/3} \phi_i^{5/3} \leq \rho_{\max}^{1/3} \quad (35)$$

and:

$$\rho_{\min}^{1/3} \leq \left(\sum_{i=1}^n \frac{\phi_i}{\rho_i} \right)^{-1/3} \leq \rho_{\max}^{1/3} \quad (36)$$

Using Equation (33), the binding energy is therefore bounded by:

$$\frac{3G}{5} \left(\frac{4\pi}{3} \right)^{1/3} M^{5/3} \rho_{\min}^{1/3} (\beta + \alpha n^{-2/3}) \leq E_p \leq \frac{3G}{5} \left(\frac{4\pi}{3} \right)^{1/3} M^{5/3} \rho_{\max}^{1/3} (\alpha + \beta) \quad (37)$$

These bounds are independent of the specific stellar fractions. They involve only the total mass, the number of stellar classes, and the minimum and maximum intrinsic densities. The upper bound is attained when all the mass belongs to the densest class.

The lower bound is generally conservative, however, because the two contributions need not reach their respective minima for the same stellar-population mixture.

4.2 Tight bounds from convex optimization

Tighter bounds can be obtained by optimizing the population-dependent part of Equation (33) directly over the stellar-population simplex. The binding energy can be written as:

$$E_p = M^{5/3} K \left[\alpha \sum_{i=1}^n \rho_i^{1/3} \phi_i^{5/3} + \beta \left(\sum_{i=1}^n \frac{\phi_i}{\rho_i} \right)^{-1/3} \right], \quad K = \frac{3G}{5} \left(\frac{4\pi}{3} \right)^{1/3} \quad (38)$$

with $\phi_i \geq 0$ and $\sum_i \phi_i = 1$. For $\alpha, \beta \geq 0$, the population-dependent function is convex over the simplex. The first contribution is a non-negative weighted sum of the convex functions $\phi_i^{5/3}$, while the second follows from applying the convex function $x^{-1/3}$ to a positive affine function of ϕ .

The exact upper bound is therefore reached at a vertex of the simplex. Assigning the entire mass to class j gives $E_p = M^{5/3} K (\alpha + \beta) \rho_j^{1/3}$. The upper bound is thus:

$$E_{\max} = M^{5/3} K (\alpha + \beta) \rho_{\max}^{1/3} \quad (39)$$

The exact lower bound is the convex optimization problem:

$$E_{\min} = M^{5/3} K \min_{\phi_i \geq 0, \sum_i \phi_i = 1} \left[\alpha \sum_{i=1}^n \rho_i^{1/3} \phi_i^{5/3} + \beta \left(\sum_{i=1}^n \frac{\phi_i}{\rho_i} \right)^{-1/3} \right] \quad (40)$$

Because the objective is convex over the unit simplex, any solution that satisfies the Karush–Kuhn–Tucker conditions is necessarily the global minimum. The lower bound can therefore be determined numerically without the ambiguity of competing local minima.

Two limiting cases are analytical. For $\beta = 0$, minimization gives $\phi_i^* = \rho_i^{-1/2} / \sum_j \rho_j^{-1/2}$ and:

$$E_{\min} = M^{5/3} K \alpha \left(\sum_{i=1}^n \rho_i^{-1/2} \right)^{-2/3} \quad (41)$$

For $\alpha, \beta > 0$, only the lower bound generally requires a numerical evaluation. Convexity guarantees that the resulting minimum is the global one within the adopted stellar-population simplex, while the upper bound remains exact and analytical.

5 Realistic bounds from stellar-population constraints

The loose and tight bounds derived above span the stellar-population space without requiring prior knowledge of the expected mass fractions. More realistic limits can be obtained by restricting that space around representative stellar populations and their associated uncertainties.

For each reference population, let f_i denote the central mass fraction of stellar class i and s_i its adopted 1σ uncertainty. Six populations are considered here, Generic, Kroupa, Early, Intermediate, Late, and Irregular. Their fractions and uncertainties are summarized in Table 3.

These reference populations are synthetic mixtures rather than direct measurements of individual galaxies. They are designed to capture broad trends expected from the stellar initial mass function, stellar evolution, and population synthesis, together with the systematic changes in stellar content and star formation observed along the galaxy morphological sequence [2, 4, 13, 17, 18]. Their purpose is therefore not to provide unique decompositions of individual galaxies, but to define physically plausible regions of stellar-population space and the uncertainties associated with them.

The class densities are treated in the same spirit. They represent characteristic mean densities rather than universal values assigned to each stellar type and are based on representative stellar masses and radii. Empirical mass–radius relations guide the values adopted for main-sequence and evolved stars, while the densities of compact remnants follow the characteristic mass–radius ranges of white dwarfs and neutron stars [11, 12, 19, 27].

Table 3: Reference stellar-population mass fractions and adopted 1σ uncertainties. Values are expressed as percentages of the total population mass.

Class	Generic	Kroupa	Early	Intermediate	Late	Irregular
BH	1.00 ± 0.50	10.0 ± 3.0	1.20 ± 0.45	1.00 ± 0.30	0.70 ± 0.20	0.50 ± 0.20
NS	0.30 ± 0.10	5.00 ± 1.50	0.45 ± 0.20	0.30 ± 0.10	0.20 ± 0.08	0.20 ± 0.08
WD	2.20 ± 0.70	20.0 ± 4.0	5.00 ± 1.30	2.50 ± 0.60	1.50 ± 0.50	1.00 ± 0.40
RG	0.20 ± 0.10	1.00 ± 0.50	1.70 ± 0.575	0.30 ± 0.10	0.20 ± 0.08	0.10 ± 0.05
M	55.0 ± 10.0	38.0 ± 6.0	53.0 ± 5.5	55.0 ± 6.0	55.5 ± 6.0	56.0 ± 7.0
K	16.0 ± 6.0	16.0 ± 3.0	18.5 ± 3.5	17.0 ± 4.0	15.5 ± 4.0	14.0 ± 4.5
G	10.5 ± 5.0	5.00 ± 1.50	10.75 ± 2.50	10.5 ± 3.0	10.0 ± 3.0	9.00 ± 3.00
F	5.50 ± 3.00	2.50 ± 0.80	4.75 ± 1.80	5.50 ± 2.00	6.00 ± 2.20	6.00 ± 2.50
A	5.00 ± 2.00	1.00 ± 0.40	2.50 ± 1.10	4.00 ± 1.20	5.00 ± 1.50	5.50 ± 1.80
B	3.00 ± 1.00	1.00 ± 0.40	1.40 ± 0.70	2.50 ± 0.70	3.00 ± 1.00	4.00 ± 1.20
O	1.30 ± 0.50	0.50 ± 0.30	0.75 ± 0.40	1.40 ± 0.40	2.40 ± 0.60	3.70 ± 0.80

The intrinsic mean densities assigned to the stellar classes are listed in Table 4. They remain fixed across all reference populations, with only the corresponding mass fractions and uncertainties allowed to vary.

Black holes are treated separately because no additional potential-energy contribution is assigned to them in the present energy accounting. Consider two bound

subsystems A and B . Before either collapses, their total contribution is $P(A) + P(B) + P(AB)$, where $P(A)$ and $P(B)$ describe the internal binding of the two subsystems and $P(AB)$ the contribution associated with their mutual aggregation.

If A collapses first, the contributions already associated with A and with the aggregation of A and B are incorporated into its black-hole mass-energy, giving $M_A = P(A) + P(AB)$ and $M_B = P(B)$. If B collapses first, the same total is recovered with $M_A = P(A)$ and $M_B = P(B) + P(AB)$.

The subsequent merger therefore gives $M_A + M_B = P(A) + P(B) + P(AB)$ regardless of the order of collapse. The merger does not generate a new potential-energy contribution, since the gravitational energy accumulated during the previous stages is already contained in the black-hole masses. Introducing an additional term at this stage would count the same contribution twice. Black holes are therefore excluded from the density-dependent potential-energy calculation.

This treatment is also consistent with the no-hair theorem. Once a collapsed subsystem has settled into a stationary black hole, its external gravitational field is specified by its mass, angular momentum, and charge. The internal structure of the original stellar system no longer remains as an independent degree of freedom from which an additional potential-energy contribution could be constructed. In particular, the binding energy associated with the finite density and spatial organization of the extended stellar subsystem has no distinct counterpart after collapse.

The black hole can still interact gravitationally with external bodies, but these interactions belong to the relation between separate systems rather than to its internal structure. They correspond to the inter-system terms considered in Section 2, which Equation (10) shows to become negligible under the scale separation relevant here. Excluding black holes from the density-dependent term therefore does not amount to neglecting their gravitational interactions. It instead distinguishes the internal binding energy already incorporated into the black-hole mass from the subdominant interaction energy that remains between distinct systems.

Table 4: Intrinsic mean densities adopted for the stellar population classes.

Class	BH	NS	WD	RG	M	K
ρ_i (kg m ⁻³)	–	4.0×10^{17}	1.0×10^9	5.0×10^{-4}	9.0×10^3	2.4×10^3
Class	G	F	A	B	O	
ρ_i (kg m ⁻³)	1.41×10^3	1.06×10^3	2.9×10^2	6.3×10^1	5.2×10^1	

Let I denote the classes with $\rho_i > 0$ and let $S = \sum_{i \in I} f_i$. The fractions and uncertainties used in the binding-energy calculation are therefore normalized over these classes as $\bar{\phi}_i = f_i/S$ and $\sigma_i = s_i/S$.

For a chosen uncertainty width $k \geq 0$, each stellar fraction is restricted to:

$$\phi_i^- = \max(0, \bar{\phi}_i - k\sigma_i), \quad \phi_i^+ = \min(1, \bar{\phi}_i + k\sigma_i) \quad (42)$$

while conservation of the total stellar mass requires $\sum_{i \in I} \phi_i = 1$. The admissible population domain is therefore:

$$\mathcal{P}_k = \left\{ \phi : \phi_i^- \leq \phi_i \leq \phi_i^+, \quad \sum_{i \in I} \phi_i = 1 \right\} \quad (43)$$

which is non-empty only when $\sum_i \phi_i^- \leq 1 \leq \sum_i \phi_i^+$.

As in the preceding bounds, the dependence on the total mass can be fully separated from the population-dependent factor. Using the constant K defined in Equation (38), the binding energy can be written as:

$$E_p = M^{5/3} K \left[\alpha \sum_{i \in I} \rho_i^{1/3} \phi_i^{5/3} + \beta \left(\sum_{i \in I} \frac{\phi_i}{\rho_i} \right)^{-1/3} \right] \quad (44)$$

The quantity constrained by the stellar population is independent of the total mass. We therefore define $P(\phi; \alpha, \beta) = E_p / M^{5/3}$. The realistic population bounds are then obtained by minimizing and maximizing P over the admissible domain $\mathcal{P}_k$:

$$P_{\min}^{(k)} = \min_{\phi \in \mathcal{P}_k} P(\phi; \alpha, \beta), \quad P_{\max}^{(k)} = \max_{\phi \in \mathcal{P}_k} P(\phi; \alpha, \beta) \quad (45)$$

and the corresponding binding-energy interval for a system of mass M is simply $M^{5/3} P_{\min}^{(k)} \leq E_p \leq M^{5/3} P_{\max}^{(k)}$.

For $\alpha, \beta \geq 0$, the population function remains convex over the restricted domain $\mathcal{P}_k$. Its minimum can therefore be obtained by convex minimization within the permitted fraction ranges. The maximum, by contrast, occurs at an extreme point of the constrained polytope, where all but at most one fraction take one of their allowed limiting values.

Because $\mathcal{P}_k$ occupies only a restricted part of the full stellar-population simplex, the resulting realistic bounds necessarily remain within the tight bounds. It therefore follows that:

$$P_{\min}^{\text{tight}} \leq P_{\min}^{(k)} \leq P(\bar{\phi}) \leq P_{\max}^{(k)} \leq P_{\max}^{\text{tight}} \quad (46)$$

At $k = 0$, the admissible domain collapses to the reference population, and the lower and upper bounds therefore become identical. As k increases, the permitted population space broadens progressively. The default choice $k = 3$ thus allows each stellar fraction to vary by as much as three adopted standard deviations from its reference value, while still enforcing non-negativity, the prescribed interval limits, and conservation of the total mass.

For the binding-energy equation derived here, $\alpha = \beta = 1$. The resulting bounds therefore represent the range of normalized binding energy compatible with a given stellar-population family, rather than with the full set of mathematically admissible mixtures.

6 Treatment of galactic gas and clumping

The preceding derivation extends directly to baryonic matter whose mass contributes coherently to the same gravitational density distribution. In the weak-field regime, the gravitational contribution is insensitive to whether the matter is stellar or gaseous, as long as both components trace the same smoothed density profile. Gas distributed in this manner can therefore be incorporated into the collective mass and contributes according to the full $M^{5/3}$ scaling obtained above.

Within this construction, the exponent $5/3$ represents the limiting contribution of the gas. It is reached when the gas participates fully in the same large-scale gravitational distribution as the stellar component. If some of the gas instead condenses into distinct clumps, part of its binding energy becomes associated with those individual structures and is no longer entirely contained in the collective contribution. For gas divided into clumps of masses m_a with $\sum_a m_a = M_g$, convexity gives:

$$\sum_a m_a^{5/3} < \left(\sum_a m_a \right)^{5/3} = M_g^{5/3} \quad (47)$$

For any appreciable degree of fragmentation, the collective $5/3$ contribution is reduced. Its maximum is attained when the gas behaves as a single component tracing the stellar distribution. As fragmentation increases, part of the gaseous mass is removed from this collective contribution and instead enters through the separate binding energies of the individual clumps. Since gaseous structures typically have much lower intrinsic densities than stellar objects, these terms are further weakened by the $\rho^{1/3}$ dependence of the binding-energy equation.

The effect of this fragmentation can be represented phenomenologically through an effective mass exponent $\gamma \leq 5/3$. The limit $\gamma = 5/3$ corresponds to gas that follows the stellar distribution without an additional correction for clumping. Progressively smaller values describe increasing fragmentation, for which a greater fraction of the gas contributes through distinct low-density structures. The parameter γ can therefore be viewed as a measure of the effective gravitational coherence of the gaseous component rather than as a change to the underlying $5/3$ scaling.

The normalized mass relation used in the empirical analysis can therefore be written as $M_{\text{BE}}/M_{\text{bar}}^\gamma$, with $5/3$ defining the maximum-efficiency limit. Values below $5/3$ indicate that an increasing share of the gaseous gravitational energy is associated with distinct clumps rather than the collective stellar distribution.

This interpretation naturally leads to an evolutionary expectation. Young, gas-rich galaxies may lie closer to the $5/3$ limit if much of their gas remains diffuse and follows the global baryonic distribution. In older systems, more of the original gas has been consumed, while the remaining material has had more time to settle into gravitationally bound and spatially distinct structures. Such systems would therefore be expected to require a stronger clumping correction and to favor a lower effective exponent.

7 Empirical validation with SPARC rotation curves

The first empirical test uses the SPARC rotation-curve sample [20]. At each measured radius r , the enclosed baryonic mass $M_{\text{bar},r}$ is evaluated together with the additional dynamical mass $M_{\text{drk},r}$ required to reproduce the observed rotation velocity. The relation tested is $M_{\text{drk},r} = E_{p,r}/c^2$.

The theoretically derived exponent $5/3$ is not assumed in this analysis. Instead, the mass exponent γ is allowed to vary freely over the broad interval $[-5, 5]$. For each trial value, every radial measurement is written as $M_{\text{drk},r}/M_{\text{bar},r}^\gamma$ and tested against the binding-energy bounds derived above.

A radial point is considered compatible when it lies between the corresponding lower and upper limits. The galaxy-level criterion is stricter, since all measured radii of a given galaxy must satisfy the bounds for the same exponent. A single incompatible point is therefore sufficient to reject that galaxy.

For the population-specific test, the admissible stellar-population bounds also depend on galactocentric radius. At $r \geq 10$, kpc, each point is tested against the specialized bounds associated with the galaxy’s morphological population. At smaller radii, a point is also accepted if it falls within the Kroupa bounds, accounting for the more Kroupa-like stellar population expected toward the inner regions. A galaxy is retained only if every radial point satisfies the applicable criterion.

7.1 Localization of the mass exponent

What stands out most clearly from the exponent scan is the narrow localization of the compatible solutions. Although γ was explored over the full interval $[-5, 5]$ in steps of 0.01, solutions are found only within a limited range around the theoretically derived value $5/3 \simeq 1.667$. Outside this region, no radial point satisfies the bounds.

For the full SPARC sample using the tight mathematical bounds (Table 5), the first compatible point occurs at $\gamma = 1.45$ and the last at $\gamma = 1.83$. Applying the galaxy-level criterion narrows this interval further, with compatible galaxies found only for $1.51 \leq \gamma \leq 1.76$. Over the remainder of the explored range, from -5 to 5 , no galaxy satisfies the criterion.

This test therefore goes beyond identifying a favorable exponent within a narrow interval chosen around $5/3$. Across most of the explored domain, no compatible solution is found at all. The admissible region emerges directly from the SPARC data and remains centered near the theoretical $5/3$ scaling for both the unrestricted and population-specific bounds. The transition is also remarkably sharp. Compatibility remains at zero over most of the scanned range, rises to 99.43% of galaxies at $\gamma = 1.62$, and falls back to zero by $\gamma = 1.77$. The theoretical value $5/3 \simeq 1.667$ lies well within this narrow empirical interval.

The same analysis was then carried out separately for each morphological subpopulation using its corresponding population bounds. Early galaxies were tested against

Table 5: Compatibility of the complete SPARC sample with the tight mathematical bounds near the narrow region selected from the full interval $\gamma \in [-5, 5]$.

γ	Points	Points (%)	Galaxies	Galaxies (%)
-5.00-1.44	0	0.00	0	0.00
1.45	1/3256	0.03	0/175	0.00
1.50	362/3256	11.12	0/175	0.00
1.55	2707/3256	83.14	94/175	53.71
1.58	3232/3256	99.26	160/175	91.43
1.60	3248/3256	99.75	169/175	96.57
1.61	3253/3256	99.91	172/175	98.29
1.62	3255/3256	99.97	174/175	99.43
1.63	3251/3256	99.85	170/175	97.14
1.65	3230/3256	99.20	163/175	93.14
1.67	3049/3256	93.64	143/175	81.71
1.70	1766/3256	54.24	101/175	57.71
1.75	227/3256	6.97	5/175	2.86
1.76	118/3256	3.62	2/175	1.14
1.77	51/3256	1.57	0/175	0.00
1.83	3/3256	0.09	0/175	0.00
1.84-5.00	0	0.00	0	0.00

the Early bounds, Intermediate galaxies against the Intermediate bounds, Late galaxies against the Late bounds, and Irregular galaxies against the Irregular bounds. As shown in Table 6, these population-specific tests produce substantially narrower admissible intervals than the Tight bounds applied to the full sample. The intervals containing complete galaxy solutions are reduced to widths of only 0.08–0.13, while the preferred exponents remain confined to $\gamma = 1.62$ – 1.66 , again very close to the theoretical value $5/3$.

Table 6: Localization of compatible solutions within the exponent interval $\gamma \in [-5, 5]$. The intervals indicate the complete range over which at least one point or one entire galaxy satisfies the corresponding bounds.

Population bounds	Points non-zero	Galaxies non-zero	Best γ
Tight, all galaxies	1.45–1.83	1.51–1.76	1.62
Early	1.50–1.72	1.56–1.64	1.62
Intermediate	1.51–1.75	1.56–1.69	1.63
Late	1.56–1.76	1.58–1.70	1.66
Irregular	1.55–1.79	1.59–1.72	1.66

The population-specific results show a systematic shift in the preferred exponent across morphological classes. Late and Irregular galaxies reach their highest compatibility at $\gamma = 1.66$, essentially matching the theoretical value $5/3$, whereas Intermediate and Early galaxies peak at $\gamma = 1.63$ and $\gamma = 1.62$, respectively. This progression is qualitatively consistent with the gas-clumping interpretation discussed in Section 6,

although the exponent localization alone does not provide a direct measure of the clumping contribution.

The breadth of the scan also offers a simple measure of how tightly the empirical solution is localized. The interval from -5 to 5 was sampled in steps of 0.01 , corresponding to approximately one thousand tested values. For the full sample and the Intermediate population, the region surrounding the maximum spans eleven tested exponents. Under a uniform-placement interpretation, the chance that the independently derived theoretical value $5/3 \simeq 1.66$ would fall within such a narrow region is therefore of order $11/1000$, or less than $1/90$. For the Late and Irregular populations, the maximum occurs directly at $\gamma = 1.66$. Under the same interpretation, allowing for the rounding of $5/3$ to either 1.66 or 1.67 , agreement with the observed maximum corresponds to a probability of approximately $1/500$.

These estimates are not formal statistical p -values, since neighboring exponent values and the different galaxy populations are not independent. They instead provide a simple measure of the geometric localization of the empirical solution within the deliberately broad interval explored. In every case, the independently predicted value lies within the narrow admissible region, and for both Late and Irregular galaxies it coincides with the maximum, providing further support for the predicted $5/3$ scaling.

7.2 Projection into the stellar-population space

The exponent scan constrains the mass scaling, whereas the normalization provides an independent test of the binding-energy prediction. Dividing the inferred missing mass by M_{bar}^γ removes the dominant dependence on the absolute baryonic mass and places each radial measurement in the mass-independent stellar-population space derived above.

For the tight mathematical bounds, the full admissible population space spans only about seven orders of magnitude. The specialized bounds associated with individual morphological populations are considerably narrower, extending over only about two orders of magnitude. Each SPARC radial measurement must therefore fall within a restricted interval set by the stellar-population model rather than by the overall mass scale of the galaxy.

The SPARC galaxies were projected as shown in Figure 1 using a single common exponent of 1.638 . This value maximizes the number of terminal galaxy points contained within the combined Generic and Kroupa plausible corridor. In this projection, the Generic+Kroupa band spans approximately three to four orders of magnitude, leaving a relatively narrow region in which the radial galaxy profiles can be compared.

With this single exponent, the terminal points of all SPARC galaxies fall within the restricted region corresponding in the model to the stellar populations associated with these systems. The allowed regions can be interpreted as population gradients, with less dense stellar components toward the lower part of the diagram and progressively denser components toward the upper part.

Several of these gradients overlap and correspond to the different galactic super-types. Their upper limits are indicated in Figure 1, while their lower limits lie

The relation can also be tested directly from the observed galaxy masses. Each galaxy is characterized by its measured $(M_{\text{bar}}, M_{\text{dark}})$ pair. To test whether the projection depends on the actual association between these two quantities, the observed values are retained while their galaxy-by-galaxy pairing is randomly permuted. If the correspondence contains no genuine structure, the randomized pairs should reproduce a similar level of compatibility with the population space.

Under this null hypothesis, the actual pairing between M_{bar} and M_{dark} should not be necessary to populate the admissible region, and randomized galaxy pairs should perform comparably. There would likewise be no particular reason for the receiving space to correspond to physically motivated stellar populations rather than arbitrary combinations of stellar fractions.

The permutation test strongly rejects this possibility. With the theoretical exponent fixed at $5/3$, 104 of the 175 observed galaxy pairs fall within the Generic population bounds, corresponding to 59.4% of the sample. After randomly permuting the association between M_{bar} and M_{dark} , the mean number of compatible galaxies falls to 54.0, or 30.9%.

None of the 10^9 random permutations reaches the observed value. The most favorable randomized configuration contains only 83 galaxies within the admissible region, still well below the 104 obtained from the actual pairings. The empirical probability is therefore below the resolution of the test, $p < 10^{-9}$, corresponding to a one-sided Gaussian-equivalent significance greater than approximately 6σ .

Because the permutation preserves the observed distributions of both M_{bar} and M_{dark} while changing only their galaxy-by-galaxy association, the result cannot be attributed to the marginal mass distributions alone. The compatibility depends specifically on the observed pairing between baryonic and dark mass.

8 Comparison with GALEX and SDSS Photometry

To associate each dynamical position with a stellar population, we use a simple multi-scale greedy reconstruction of the stellar fractions. The procedure starts from the generic reference population given in Table 3 and iteratively adjusts the fractions until the normalized mass inferred from the galaxy dynamics is reproduced.

For a stellar population described by the fraction vector $\phi = (\phi_1, \dots, \phi_n)$, each trial configuration defines a predicted normalized mass $\mathcal{M}_{\text{pred}}(\phi)$. The optimization seeks the configuration that minimizes the relative difference between this prediction and the normalized mass inferred from the observations:

$$\mathcal{E}_{\text{norm}}(\phi) = \frac{|\mathcal{M}_{\text{pred}}(\phi) - \mathcal{M}_{\text{obs}}|}{\mathcal{M}_{\text{obs}}} \quad (48)$$

At each iteration, the stellar fractions are perturbed locally around the current solution and then renormalized. The change that produces the largest reduction in $\mathcal{E}_{\text{norm}}$ is retained, after which the perturbation amplitude is gradually decreased until

Table 7: Pearson and Spearman correlations between predicted and observed GALEX/SDSS colors for $\alpha = \beta = 1$.

Method	Color	N	r_P	σ_P	ρ_S	σ_S
Generic	FUV-NUV	136	0.316	3.75	0.388	4.67
	$r - z$	129	0.258	2.95	0.291	3.34
	$g - r$	128	0.426	5.02	0.502	6.04
	NUV- r	119	0.447	5.10	0.542	6.36
By type	FUV-NUV	136	0.492	6.08	0.586	7.50
	$r - z$	129	0.508	6.15	0.502	6.06
	$g - r$	128	0.788	11.04	0.789	11.05
	NUV- r	119	0.806	11.05	0.809	11.13
Photometry-guided						
By type	FUV-NUV	136	0.608	7.85	0.683	9.16
	$r - z$	129	0.522	6.34	0.609	7.66
	$g - r$	128	0.799	11.30	0.881	13.73
	NUV- r	119	0.817	11.32	0.889	13.50
Mass-photometry	FUV-NUV	–	–	–	–	–
	$r - z$	122	0.365	4.13	0.476	5.54
	$g - r$	121	0.570	6.82	0.683	8.64
	NUV- r	119	0.486	5.60	0.521	6.08

no further improvement is found. For each galaxy, this procedure reconstructs a stellar population whose gravitational estimator matches its position in normalized-mass space. Throughout the photometric reconstruction, the mass exponent is fixed at the common best-fit value of 1.638 for all galaxies (see Figure 1).

8.1 Dependence on the stellar-population prescription

We first evaluate the photometric reconstruction using the same Generic reference population for every galaxy. The analysis is then repeated with morphology-dependent reference populations, assigning each galaxy to the Early, Intermediate, Late, or Irregular population corresponding to its morphological supertype (Table 3).

A third configuration, referred to as Photometry-guided, starts from the same morphology-dependent assignment but allows each galaxy to move within its population space. The individual mass exponent is then adjusted greedily to maximize the photometric correlations.

The resulting population-based predictions are compared with independent empirical mass-photometry relations. The $g - r$ and $r - z$ colors are estimated from the stellar-mass relations of [6], while NUV- r is derived from the H,I gas-fraction plane of [5], which relates atomic gas fraction, stellar surface mass density, and NUV- r color. No comparable empirical relation is used for FUV-NUV.

Table 7 compares the Pearson and Spearman correlations obtained with the four prescriptions. Even the single Generic population produces positive correlations for all four colors, with the strongest relations appearing for $g - r$ and NUV- r . Introducing morphology-dependent populations substantially strengthens all four correlations. For $g - r$ and NUV- r , both Pearson and Spearman significances exceed 11σ . As expected,

the Photometry-guided By-type configuration produces the strongest correlations, with Spearman significances above 13σ for both $g-r$ and $\text{NUV}-r$.

The independent mass-photometry models also yield positive correlations for every color for which a calibration is available. These correlations are generally stronger than those obtained with the single Generic population, but weaker than those from the morphology-dependent reconstruction. No comparable empirical estimator is available for FUV-NUV, so the corresponding entry is left undefined. The resulting strengths and scatter are consistent with the empirical mass-photometry relations reported by [6] and [5].

Beyond supporting the model through these correlations, the reconstruction provides a new photometric estimator whose performance is comparable to that of relations developed specifically for photometric prediction, even though photometry plays no role in the construction of the underlying model. The distinction is important. Conventional empirical relations rely primarily on baryonic mass-to-light properties, whereas the present approach links the baryonic-to-dark mass ratio, and therefore the galaxy dynamics, directly to photometric properties. This yields a direct dynamical-photometric relation that, to our knowledge, has no existing counterpart in the literature and represents a distinct result of the analysis.

8.2 Statistical robustness of the photometric relation

The four prescriptions—Generic, By type, Photometry-guided, and Mass-photometry—were subjected to the same statistical robustness tests. For the Generic and By-type reconstructions, we used $\alpha = \beta = 1$ together with the common exponent 1.638. The Photometry-guided reconstruction begins from the same By-type configuration but allows the exponent of each galaxy to be adjusted greedily to optimize the photometric correlations. For each prescription, we evaluated 10^6 bootstrap realizations, leave-one-out jackknife estimates, and 10^8 random permutations. The correlations remain robust across all four cases, with stable bootstrap intervals, rare or absent sign reversals, and strong permutation significances. For the Photometry-guided configuration, however, this robustness characterizes the optimized solution itself and should not be interpreted as an independent photometric validation.

The inclination tests show that the correlations cannot be explained simply by viewing geometry. Significant relations persist after dividing the sample by inclination for all four prescriptions. Even for the Generic reconstruction, the Spearman significance reaches 4.15σ for $g-r$ and 4.58σ for $\text{NUV}-r$ in the high-inclination subsample. In the intermediate-inclination range, the $g-r$ relation reaches the 5.73σ permutation limit for the By-type, Photometry-guided, and Mass-photometry prescriptions. The Photometry-guided $\text{NUV}-r$ relation reaches the same limit in this inclination range.

The morphology tests reveal a different pattern. For the Generic, By-type, and Mass-photometry prescriptions, dividing the sample into Early, Intermediate, Late, and Irregular galaxies substantially weakens the correlations. Within individual morphological classes, the permutation significances remain below 1.92σ for Generic and 1.90σ for By type. The Mass-photometry prescription reaches 3.53σ for $g-r$ and

2.80σ for $\text{NUV}-r$ in the Intermediate subsample. The Photometry-guided configuration behaves differently, retaining strong within-class correlations after the individual galaxy positions have been optimized using photometry. Within the Intermediate population, both $g-r$ and $\text{NUV}-r$ reach the 5.73σ permutation limit. This difference is expected because the Photometry-guided configuration explicitly uses photometric information to refine the galaxy positions, whereas the other prescriptions do not.

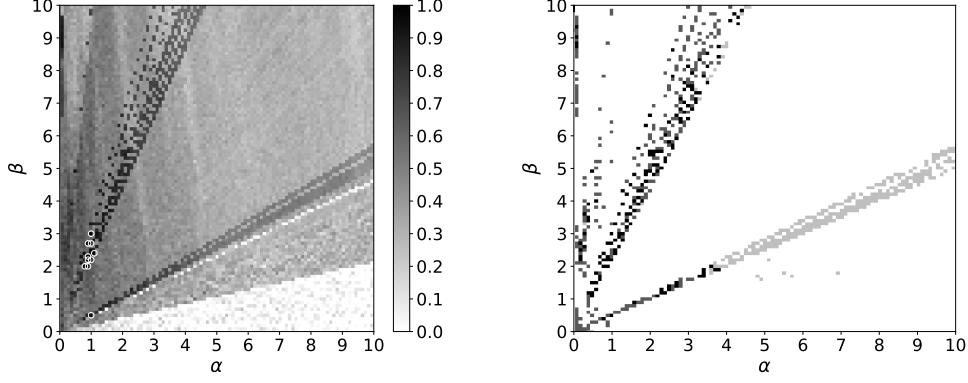
Taken together, these results show that projecting normalized dark mass into population space with a single global exponent is already sufficient to recover photometric relations comparable to, and in some cases stronger than, empirical mass–photometry prescriptions developed specifically for this purpose. The robustness tests further indicate that much of this predictive power comes from the relative placement of the morphological supertypes within population space. When galaxies are considered only within a single supertype, the correlations generally weaken, suggesting that the exact galaxy-by-galaxy ordering contributes less than the separation between the super-types themselves. A similar morphology dependence is also present in the empirical mass–photometry relations. The dynamical–population correspondence is nevertheless highly constrained, since an unrelated association between the dynamical projection and population space would be extremely unlikely to reproduce the observed photometric ordering.

The Photometry-guided optimization provides a complementary test of the mapping between normalized mass and population space. By allowing the individual galaxy exponents to vary only within the narrow interval $\sim 1.605\text{--}1.666$, it identifies a projection that reproduces the photometric ordering with a rejection of the corresponding null hypothesis above 5σ . This demonstrates that a statistically valid mapping from normalized mass into population space can be established. The narrow range of required exponents further shows that this mapping can be obtained without arbitrary displacements across population space, but only through limited adjustments around the common exponent.

8.3 Photometric exploration of the estimator

The binding-energy estimator introduced in Equations (30) and (34) contains two coefficients with distinct roles. The coefficient α controls the contribution from the homogeneous stellar classes, whereas β governs the global aggregation term. For each pair (α, β) , stellar populations are reconstructed from the normalized-mass constraint and their predicted GALEX and SDSS colors are compared with the observations. The Pearson and Spearman significances obtained for $\text{FUV}-\text{NUV}$, $\text{NUV}-r$, $g-r$, and $r-z$ are then combined into a single photometric score. This provides an independent way to test the stellar populations generated by different combinations of the two coefficients.

The photometric score extends the exponent tests previously applied to the mass term to the parameters that determine the population-space relation itself. Variations in α and β produce only modest shifts in logarithmic population space, whereas changes in the population exponents can modify the admissible width by several orders



(a) Normalized photometric score over the (α, β) grid. Darker shades correspond to higher scores, while the white symbols identify the ten highest-scoring solutions.

(b) Preferred regions of parameter space. Light gray marks the five best values of β for each fixed α , dark gray marks the five best values of α for each fixed β , and black marks solutions selected by both criteria.

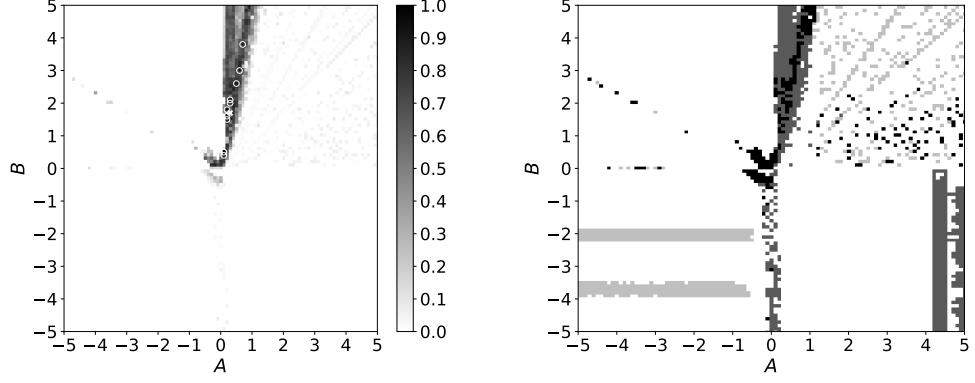
Fig. 2: Photometric exploration of the two-term binding-energy estimator. The preferred solutions are strongly concentrated around $\alpha = 1$, while the dependence on β remains broader and extends on both sides of $\beta = 1$.

of magnitude. The population-band criterion is therefore weakly discriminating in either case, while the photometric score remains sensitive to the resulting changes in stellar populations.

Figure 2 shows the exploration of the (α, β) parameter space. In panel (a), the gray scale represents the normalized photometric score across the full grid, with darker regions corresponding to higher scores. The white circles identify the ten highest-scoring configurations. Panel (b) instead emphasizes the location of the preferred solutions independently of their absolute score. For each fixed α , the five best values of β are shown in light gray, while for each fixed β , the five best values of α are shown in dark gray. Configurations selected by both criteria are shown in black.

We then repeat the same photometric analysis after replacing the exponents $1/3$ and $5/3$ in the α term of Equation (34) with two free parameters, denoted A and B . The population contribution to the normalized binding-energy estimator therefore becomes proportional to $\sum_i \rho_i^A \phi_i^B$, while the global aggregation term controlled by β is left unchanged. For each pair (A, B) , stellar populations are reconstructed from the normalized-mass constraint and their predicted GALEX and SDSS colors are compared with the observations using the same procedure as in the (α, β) exploration.

Figure 3 shows the resulting (A, B) parameter space. As before, panel (a) gives the normalized photometric score across the full grid, with darker regions indicating higher scores and the ten best configurations marked by white circles. The theoretical point $(A, B) = (1/3, 5/3)$ is indicated by a cross. Panel (b) instead highlights the



(a) Normalized photometric score over the (A, B) grid. Darker shades correspond to higher scores, while the white circles identify the ten highest-scoring solutions. The white cross marks the theoretical value $(A, B) = (1/3, 5/3)$.

(b) Preferred regions of exponent space. Light gray marks the best values of B for each fixed A , dark gray marks the best values of A for each fixed B , and black marks solutions selected by both criteria.

Fig. 3: Photometric exploration of the exponents entering the α term of Equation (34). The parameters A and B replace the theoretical exponents $1/3$ and $5/3$, respectively.

preferred region independently of the absolute score. For each fixed A , the five best values of B are marked in light gray, and for each fixed B , the five best values of A are marked in dark gray. Points selected by both criteria are shown in black.

The theoretical parameters are strongly localized in both explored spaces. In the (α, β) plane, all ten highest-scoring solutions lie within the narrow interval $\alpha \in [0.8, 1.1]$. Four occur at $\alpha = 1$, four at $\alpha = 0.9$, one at $\alpha = 1.1$, and one at $\alpha = 0.8$, placing the theoretical value $\alpha = 1$ near the center of the concentration. Along the β direction, the theoretical value $\beta = 1$ lies between one of the ten best solutions at $(1, 0.5)$ and the main concentration of preferred solutions near $(1, 2.5)$. These intervals cover approximately 4% and 25% of the explored α and β ranges. Under a uniform-placement interpretation, the probability of a randomly located point falling within both intervals is therefore about 1%. The population exponents show a similarly restricted localization. The theoretical point $(A, B) = (1/3, 5/3)$ lies within the only clearly correlated region of the (A, B) plane, which occupies about 2.5% of the explored area, and directly beside two of the ten highest-scoring solutions. Under the same uniform-placement interpretation, the probability of falling within this region by chance is therefore about 2.5%, while the more restrictive nearest-neighbor criterion gives a probability below 1%.

Taken together, the photometric analysis independently localizes the theoretically derived values of α , β , A , and B within small preferred regions of their respective parameter spaces. Using the more restrictive localization criteria, the combined chance probability is of order 10^{-4} .

9 Converging evidence for a genuine projection

The hypothesis tested throughout this analysis is that the binding-energy equation defines a genuine projection of normalized dark mass into stellar-population space. The Photometry-guided reconstruction provides the most direct test of this idea. Allowing the exponent of each galaxy to vary only within the narrow interval 1.605–1.666, close to the theoretical value $5/3$, produces a galaxy-by-galaxy projection whose photometric correspondence rejects the null hypothesis at more than 5σ .

Even without these corrections, however, a single common exponent already projects the full sample well enough to recover photometric relations comparable to, and in some cases stronger than, empirical mass–photometry relations developed specifically for that purpose. This global correspondence between two independently constructed spaces therefore provides an additional restrictive constraint. Moreover, all 175 terminal galaxy points fall within the admissible population bounds, corresponding to a chance probability below $(1/2)^{175} \simeq 2 \times 10^{-53}$ even under the conservative assumption of equal probabilities of falling inside or outside the allowed region.

The correspondence also depends on the actual structure of the galaxy population. It is recovered only when each galaxy retains its observed association between baryonic and dark mass. When these associations are randomized while preserving the overall distributions of both quantities, the correspondence with population space is rejected at approximately 6σ . The projection therefore depends on the galaxy-by-galaxy organization of these quantities, not simply on reproducing their observed ranges.

The algebraic structure of the energy relation can be tested through two independent aspects of the projection. Its mass dependence is constrained directly by the location of the galaxies in population space, which localizes the exponent near the theoretical value $5/3$ with an estimated chance probability of approximately 5×10^{-2} . The remaining population-space structure is constrained independently by the photometric correlations. Varying α , β , A , and B localizes the preferred solutions around their theoretical values with an estimated chance probability of approximately 10^{-4} . Together, these tests constrain both the mass scaling and the internal algebraic structure of the relation, giving a combined chance probability of approximately 10^{-7} , corresponding to more than 5σ , for the theoretical form to be recovered accidentally.

These results constrain distinct aspects of the same hypothesis: the existence of a valid galaxy-by-galaxy projection, a global projection using a single common exponent, the preservation of the observed $(M_{\text{bar}}, M_{\text{dark}})$ pairings, and the recovery of the theoretical algebraic structure of the energy equation. As a deliberately conservative estimate, assigning a chance probability of only 10^{-2} to each of these four constraints gives:

$$(10^{-2})^4 = 10^{-8}$$

This still corresponds to a significance above 5σ , so even under a substantially weakened estimate, a chance origin for the observed correspondence remains strongly disfavored.

10 Dark mass and dark energy

The galactic-scale results point toward a more speculative extension of the framework. If the effective dark mass inferred from galaxy dynamics reflects gravitational binding energy, the same hierarchical construction need not end at the scale of individual galaxies. Galaxies may themselves act as bound subsystems contributing to the gravitational energy of larger structures. Extending the argument to these scales, however, requires cosmic expansion to be taken into account and therefore calls for a cosmological treatment beyond the local weak-field description used.

10.1 Extragalactic extension and the turnaround scale

At galactic scales, characteristic velocities remain low and cosmic expansion has little influence on the internal dynamics, so the Newtonian weak-field approximation remains adequate. On sufficiently large scales, however, the expansion of the Universe begins to compete with gravitational binding. The turnaround radius provides a natural boundary between these two regimes. In a Schwarzschild–de Sitter description, its maximum value is:

$$R_{\text{ta,max}} = \left(\frac{3GM}{\Lambda c^2} \right)^{1/3} \quad (49)$$

which sets the largest scale over which a mass M can remain gravitationally bound in the presence of a positive cosmological constant [22, 26].

This provides a natural continuation of the hierarchy developed above. Stellar classes are first assembled into galaxies, which can then be combined within gravitationally bound turnaround regions. In a simple implementation, cells containing from a few galaxies to several hundred produce an extragalactic binding-energy contribution of the same order as the cosmological dark-mass component. Their populations are comparable to those expected for isolated systems, groups, and clusters. This is not intended as an independent cosmological determination, but it suggests that the hierarchical construction may remain physically relevant beyond the scale of individual galaxies.

The turnaround condition is essential because it prevents the calculation from being extended arbitrarily across the Universe. Systems lying beyond their effective gravitational domain should not contribute in the same manner as systems belonging to a common bound structure. The turnaround scale thus provides an approximate boundary between locally stored binding energy and the large-scale dynamics driven by cosmic expansion.

Extending the construction to earlier epochs introduces two simultaneous effects. First, stellar populations evolve with redshift. The lower abundance of dense stellar remnants at early times reduces the characteristic stellar density and hence the galactic binding-energy contribution. Second, the turnaround scale also evolves. At higher redshift, the effective turnaround radius is smaller, reducing both the size and mass of the regions within which galaxies can remain bound. The extragalactic contribution therefore declines even more rapidly.

Although the gravitational-energy mechanism reproduces the expected dark-mass scale at $z = 0$, this agreement progressively deteriorates toward higher redshift. The predicted galactic contribution falls to about 41% of the expected value at $z = 5$ and 11% at $z = 10$, while the extragalactic contribution declines to about 10% and 2%, respectively.

This leads to what we refer to as a cosmological hypercoincidence. Two independent physical evolutions, the changing stellar density and the evolving turnaround scale, both drive the predicted dark mass away from its present-day value. The agreement at $z = 0$ therefore appears linked to a particular cosmic epoch rather than emerging generically from the model. This tension motivates the possible exchange between gravitational and vacuum energy developed below.

10.2 Gravitational–vacuum energy exchange

A deeper connection appears if gravitational binding energy is allowed to exchange energy with the vacuum. General relativity imposes local covariant conservation, but an expanding spacetime does not generally admit a globally conserved energy associated with time-translation symmetry. The essential step is therefore to allow the effective cosmological term to vary locally in spacetime, $\Lambda = \Lambda(x^\mu)$, rather than treating it as a fixed universal constant. This removes the need to impose a separate global conservation law on the gravitational sector, where the usual Noether limitation applies. Local changes in gravitational energy can instead be compensated by corresponding changes in the vacuum contribution, while the total stress–energy remains covariantly conserved.

At the covariant level, this can be written as:

$$\nabla_\mu T^{\mu\nu} = -\frac{c^4}{8\pi G} \nabla^\nu \Lambda \quad \Longleftrightarrow \quad \nabla_\mu \left(T^{\mu\nu} + T_{(\Lambda)}^{\mu\nu} \right) = 0 \quad (50)$$

For a gravitationally bound turnaround cell followed through cosmic evolution, let $V_{\text{ta}}(t)$ denote its physical volume and define:

$$\Theta_{\text{ta}} = \frac{\dot{V}_{\text{ta}}}{V_{\text{ta}}}$$

The exchange between gravitationally stored energy and the vacuum sector may then be written as:

$$\dot{\rho}_{\text{grav}} + \Theta_{\text{ta}} \rho_{\text{grav}} = Q, \quad \dot{\rho}_\Lambda + \Theta_{\text{ta}} \rho_\Lambda = -Q \quad (51)$$

where Q represents the energy transferred between the two sectors within the same turnaround cell.

Equivalently, defining $E_{\text{grav}} = \rho_{\text{grav}} V_{\text{ta}}$ and $E_\Lambda = \rho_\Lambda V_{\text{ta}}$ gives:

$$\frac{d}{dt} (E_{\text{grav}} + E_\Lambda) = 0 \quad (52)$$

Under this interpretation, dark mass and dark energy are no longer independent components. Dark mass corresponds to gravitational energy stored in bound structures, while the vacuum sector provides the complementary cosmological contribution as this stored energy evolves through structure formation and cosmic expansion.

This possibility becomes particularly relevant because the galactic contribution to gravitational energy is not expected to remain constant with redshift. As stellar populations evolve, so does the amount of binding energy stored on galactic scales. A compensating vacuum contribution could preserve the overall gravitational energy budget across an ensemble of evolving turnaround regions, even as its partition between bound structures and the cosmological background changes.

On cosmological scales, the framework predicts a systematic redistribution of the inferred dark component between galactic and extragalactic environments. While the extragalactic contribution associated with turnaround structures remains approximately constant with redshift, the galactic contribution decreases toward high z , producing a progressive decline in $M_{\text{dark}}^{\text{gal}}/M_{\text{dark}}^{\text{ext}}$. This is consistent with a growing fraction of the effective gravitational energy being associated with group-, filament-, and circumgalactic-scale structures rather than individual galactic halos. Such a redistribution could favor larger hot-gas reservoirs in the CGM, ICM, and WHIM, reduce excessive central baryon capture and stellar production, and strengthen the meso-scale gravitational environment without disturbing the large-scale successes of Λ CDM.

The turnaround scale provides a natural interface for this exchange. It separates gravitationally bound regions, where potential energy can be stored as an effective dark mass, from the expanding background, where the vacuum contribution governs the large-scale evolution. Changes in the characteristic turnaround scale can therefore mediate the evolving balance between the two sectors.

10.3 Thermodynamic closure and a zero-energy Universe

The exchange described above suggests a broader interpretation of the cosmological energy budget. General relativity does not, in general, define a globally conserved energy in an expanding spacetime, whereas the first law of thermodynamics expresses energy conservation for a closed system. These statements are not mathematically contradictory, since they arise from different conceptual frameworks. The present hypothesis nevertheless suggests a possible connection between them when the Universe is treated as a whole.

If dark mass corresponds to gravitational potential energy, the positive energy content of the Universe includes not only the rest energy of baryonic matter but also the gravitational energy stored as bound structures form. We may therefore write:

$$E_+ = E_{\text{bar}} + E_{\text{pot}} \quad (53)$$

The thermodynamic closure proposed here assigns the cosmological vacuum sector the compensating contribution:

$$E_{\text{tot}} = E_{\text{bar}} + E_{\text{pot}} + E_{\Lambda} \quad (54)$$

The strongest form of the hypothesis is then:

$$E_{\text{tot}} = 0, \quad E_{\Lambda} = -(E_{\text{bar}} + E_{\text{pot}}) \quad (55)$$

Other contributions, such as radiation, can be included in the global energy balance without altering the conservation principle considered here. They are omitted for clarity because they do not affect the gravitational–vacuum mechanism developed in this work.

In this picture, the positive energy associated with baryonic matter and gravitational potential energy is balanced by the cosmological contribution. The total energy of the Universe is therefore zero, while its observable contents correspond to different partitions of this vanishing global balance.

This interpretation extends beyond the dark-sector problem. General relativity provides a local geometrical description of gravitation and guarantees covariant conservation, but does not generally define a conserved global energy in an expanding Universe. Thermodynamics, by contrast, motivates conservation of the total energy of a closed system. The extended conservation proposed here offers a possible connection between these two descriptions. Local evolution remains governed by general relativity, while the global energy budget is subject to an additional thermodynamic constraint.

The condition $E_{\text{tot}} = 0$ gives this picture a particularly simple form. The Universe may contain very large positive contributions from baryonic matter and gravitational energy while retaining zero net energy overall, with the cosmological contribution providing the corresponding balance. Structure formation, gravitational binding, and accelerated expansion can then be viewed as different manifestations of the same global energy budget.

This perspective also changes the interpretation of the cosmological coincidence problem. Dark mass and dark energy need not have comparable magnitudes by chance if both are governed by the same conservation condition. Their relative importance would instead reflect how the total energy budget is partitioned between gravitationally bound structures and the cosmological background at a given epoch.

The zero-energy condition remains an additional physical hypothesis and does not follow from the Einstein equations alone. Its interest lies in the possibility that the apparent absence of global energy conservation in cosmological general relativity reflects not a failure of thermodynamics, but an incomplete accounting of gravitational and vacuum contributions. If this interpretation is correct, the dark sector would provide a natural link between general relativity and the first law of thermodynamics on the scale of the Universe.

11 Discussion

The central question of this work is straightforward. If all the stars in a galaxy were brought together through an idealized perfectly inelastic aggregation involving only gravity, how much energy would be released? The quantity calculated here is intended to represent precisely this energy. It corresponds to the gravitational energy stored in the internal organization of the galaxy and potentially available through a change in configuration.

The connection with dark mass comes only afterward. Through mass–energy equivalence, physically stored gravitational energy corresponds to an effective mass contribution. If this contribution participates in the gravitational field, it should also influence spacetime curvature and the motion of test bodies. The observational question is therefore whether the gravitational effects usually attributed to an additional dark matter component could instead correspond to the mass equivalent of this binding energy.

For decades, the simplest interpretation of the missing-mass problem was to introduce an unseen matter component while leaving the laws of gravitation unchanged. This was initially economical because the missing component could plausibly have been ordinary matter not yet detected, or a new elementary-particle species. As the dominant baryonic candidates have been progressively excluded and no non-baryonic component has yet been independently identified, however, this original argument from simplicity has become less compelling.

The interpretation proposed here applies the same principle of economy differently. It requires neither a new form of matter nor a modification of the known laws of gravitation. The additional gravitating contribution instead comes from gravitational potential energy already present in the system, together with ordinary mass–energy equivalence. The missing-mass problem is thus recast not as evidence for an additional substance, but as a possible consequence of how gravitational energy is accounted for in bound systems.

The exceptionally broad range over which the construction can be established provides another reason to consider this possibility seriously. The Newtonian derivation preserves the same functional form across nearly forty orders of magnitude in mass. Its eventual breakdown at high compactness reflects the failure of the Newtonian construction rather than of the resulting mathematical structure. An independent construction based on sequential relativistic absorption reaches the same expression without extending a hypothetical Newtonian configuration beyond its domain of validity. The convergence of these two physically distinct constructions therefore supports the robustness of the relation.

A fully relativistic derivation would nevertheless remain valuable, particularly for clarifying the spacetime interpretation of the gravitational-energy contribution. It should not, however, be expected to provide a stronger validation of the equation itself. The relation is already supported by its persistence across nearly forty orders of magnitude, by the convergence of two distinct derivations, and by its agreement with

galactic observations. A covariant derivation would therefore strengthen its theoretical foundation and interpretation rather than its empirical support.

The binding-energy equation makes two distinct predictions. Its dependence on total baryonic mass follows a $M^{5/3}$ scaling, while its normalization depends on the stellar composition of the galaxy through the population fractions and stellar densities. The mass scaling was tested by replacing the theoretical exponent with a free parameter γ and comparing the rescaled dark-mass estimates with independently derived population bounds. The resulting optimum is sharply localized near $5/3$, with Late and Irregular galaxies lying closest to the theoretical value.

The population structure was tested independently by varying α , β , A , and B and measuring the resulting GALEX and SDSS photometric correlations. The strongest correlations occur near the theoretically expected values. A separate permutation test also shows that the correspondence depends on the observed galaxy-by-galaxy association between M_{bar} and M_{dark} . Randomizing these pairings while preserving both marginal mass distributions substantially destroys the agreement with population space.

A single common exponent is nevertheless sufficient to place the terminal points of all 175 galaxies within the admissible bands and to produce photometric models comparable in quality to existing empirical relations. Even under the deliberately conservative assumption that each galaxy has a probability as large as $1/2$ of falling within the allowed region by chance, the probability that all 175 do so is bounded by $(1/2)^{175} \simeq 2.1 \times 10^{-53}$. Allowing only small galaxy-by-galaxy adjustments to the exponent strengthens the GALEX and SDSS correspondence further, with the corresponding null hypothesis rejected at more than 5σ .

These tests therefore constrain several distinct aspects of the same relation: the predicted $M^{5/3}$ mass dependence, the stellar-population normalization, the observed pairing between baryonic and dark mass, the existence of a global projection using a single exponent, and the more precise galaxy-by-galaxy photometric projection obtained with only small exponent adjustments.

Taken together, they leave little room for the correspondence to arise by chance. Even when retaining only four distinct tests, and deliberately weakening each major constraint to a chance probability of just 10^{-2} , the combined probability under the null remains 10^{-8} , corresponding to a significance greater than 5σ .

The experimental tests can now be extended beyond galactic rotation curves and stellar photometry. The same relation should be confronted with independent observables probing the gravitational field in different ways, including gravitational lensing, stellar and gas dynamics outside galactic disks, satellite motions, galaxy groups, and clusters. Agreement across these regimes would test whether the same gravitational-energy contribution consistently accounts for effects currently attributed to dark mass rather than merely reproducing one class of galactic observations.

Extending the construction to cosmological scales reveals a strong redshift dependence that the galactic mechanism alone does not resolve. A possible exchange between gravitational and vacuum energy offers one route toward a unified interpretation of

the dark sector, while the zero-energy condition introduces an additional thermodynamic hypothesis. These extensions remain speculative, but they also provide further opportunities for the framework to be tested and potentially falsified.

The galactic results nevertheless show that gravitational binding energy exhibits the predicted mass dependence, normalization, and stellar-population structure required for its proposed connection with dark mass to be tested directly.

12 Conclusion

This work proposes that the mass conventionally attributed to dark matter may instead be interpreted as the mass equivalent of gravitational potential energy stored in bound structures. The resulting relation follows from standard gravitational physics and mass–energy equivalence, without requiring a new form of matter or a modification of the known laws of gravitation. Its agreement with galactic observations shows that this interpretation can reproduce both the magnitude and the structural dependence of the missing-mass phenomenon.

The framework can now be confronted with observables beyond those considered here. Gravitational lensing, external galactic dynamics, galaxy groups, clusters, and cosmological evolution provide independent regimes in which the same interpretation can be tested. If the correspondence persists across these scales, dark mass may be understood not as an additional substance, but as a manifestation of gravitational energy already contained in the structure of the Universe.

Software Availability

The C++ program used to perform all numerical calculations and generate the corresponding graphs are freely available at dark-mass-generator.sourceforge.io or at <https://doi.org/10.6084/m9.figshare.33894055>.

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References

- [1] Bartelmann M (2010) Gravitational lensing. *Classical and Quantum Gravity* 27(23):233001. <https://doi.org/10.1088/0264-9381/27/23/233001>
- [2] Blanton MR, Moustakas J (2009) Physical properties and environments of nearby galaxies. *Annual Review of Astronomy and Astrophysics* 47:159–210. <https://doi.org/10.1146/annurev-astro-082708-101734>
- [3] Brillouin L (1965) The mass of a system of potential energy. *Proceedings of the National Academy of Sciences of the United States of America* 53(3)
- [4] Bruzual G, Charlot S (2003) Stellar population synthesis at the resolution of 2003. *Monthly Notices of the Royal Astronomical Society* 344(4):1000–1028. <https://doi.org/10.1046/j.1365-8711.2003.06897.x>
- [5] Catinella B, Schiminovich D, Kauffmann G, et al (2010) The galex arecibo sdss survey – i. gas fraction scaling relations of massive galaxies and first data release. *Monthly Notices of the Royal Astronomical Society* 403(2):683–708. <https://doi.org/10.1111/j.1365-2966.2009.16180.x>
- [6] Chang R, Shen S, Hou J, et al (2006) Colour–magnitude relations of late-type galaxies. *Monthly Notices of the Royal Astronomical Society* 372(1):199–208. <https://doi.org/10.1111/j.1365-2966.2006.10826.x>
- [7] Clowe D, Bradač M, Gonzalez AH, et al (2006) A direct empirical proof of the existence of dark matter. *The Astrophysical Journal Letters* 648:L109–L113
- [8] Courteau S, et al (2014) Galaxy masses. *Reviews of Modern Physics* 86:47–119
- [9] Deur A (2021) Relativistic corrections to the rotation curves of disk galaxies. *European Physical Journal C* 81:213
- [10] Deur A, Sargent C, Terzic B (2020) Significance of gravitational nonlinearities on the dynamics of disk galaxies. *The Astrophysical Journal* 896:94
- [11] Eker Z, Bakis V, Bilir S, et al (2018) Interrelated main-sequence mass–luminosity, mass–radius, and mass–effective temperature relations. *Monthly Notices of the Royal Astronomical Society* 479(4):5491–5511. <https://doi.org/10.1093/mnras/sty1834>
- [12] Fontaine G, Brassard P, Bergeron P (2001) The potential of white dwarf cosmochronology. *Publications of the Astronomical Society of the Pacific* 113(782):409–435. <https://doi.org/10.1086/319535>
- [13] Gallazzi A, Charlot S, Brinchmann J, et al (2005) The ages and metallicities of galaxies in the local universe. *Monthly Notices of the Royal Astronomical Society* 362(1):41–58. <https://doi.org/10.1111/j.1365-2966.2005.09321.x>

- [14] Hawking SW (1968) Gravitational radiation in an expanding universe. *Journal of Mathematical Physics* 9(4):598–604
- [15] Hayward SA (1994) Quasilocal gravitational energy. *Physical Review D* 49(2):831–839. <https://doi.org/10.1103/PhysRevD.49.831>
- [16] Hayward SA (1996) Gravitational energy in spherical symmetry. *Physical Review D* 53:1938–1949
- [17] Kennicutt RC (1998) Star formation in galaxies along the hubble sequence. *Annual Review of Astronomy and Astrophysics* 36:189–232
- [18] Kroupa P (2001) On the variation of the initial mass function. *Monthly Notices of the Royal Astronomical Society* 322(2):231–246. <https://doi.org/10.1046/j.1365-8711.2001.04022.x>
- [19] Lattimer JM, Prakash M (2007) Neutron star observations: Prognosis for equation of state constraints. *Physics Reports* 442(1-6):109–165
- [20] Lelli F, McGaugh SS, Schombert JM (2016) Sparc: Mass models for 175 disk galaxies with spitzer photometry and accurate rotation curves. *The Astronomical Journal* 152(6):157
- [21] Misner CW, Sharp DH (1964) Relativistic equations for adiabatic, spherically symmetric gravitational collapse. *Physical Review* 136:B571–B576
- [22] Pavlidou V, Tomaras TN (2014) Where the world stands still: turnaround as a strong test of Λ CDM cosmology. *Journal of Cosmology and Astroparticle Physics* 2014(09):020
- [23] Persic M, Salucci P, Stel F (1996) The universal rotation curve of spiral galaxies. *Monthly Notices of the Royal Astronomical Society* 281:27–47
- [24] Rubin VC, Ford JW, Kent, Thonnard N (1980) Rotational properties of 21 sc galaxies with a large range of luminosities and radii, from ngc 4605 ($r = 4\text{kpc}$) to ugc 2885 ($r = 122\text{kpc}$). *The Astrophysical Journal* 238:471–487
- [25] Szabados LB (2009) Quasi-local energy-momentum and angular momentum in general relativity. *Living Reviews in Relativity* 12(4):4. <https://doi.org/10.12942/lrr-2009-4>
- [26] Tanoglidis D, Nesseris S, Pavlidou V (2015) Testing Λ CDM cosmology at turnaround: where to look for violations of the bound? *Journal of Cosmology and Astroparticle Physics* 2015(12):060
- [27] Torres G, Andersen J, Gimenez A (2010) Accurate masses and radii of normal stars: modern results and applications. *The Astronomy and Astrophysics Review* 18(1-2):67–126. <https://doi.org/10.1007/s00159-009-0025-1>