

# Symmetries of Natural Numbers

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## Abstract

In present paper we prove an original theorem on the natural numbers. From this theorem emerges a set of symmetries of the natural numbers. “The conjugate”, “the complementary”, “the L/R symmetry”, “the transpose”, and “the K numbers” did not exist as concepts in Number Theory. The same applies to “the octets of odd numbers”. Through these symmetries the study of the natural numbers can be done, outside the context in which they have been studied so far. One of the consequences of the theorem, but not the most important one, is an odd number factorization algorithm.

## Keywords

Symmetries, categorization of odd numbers, octets of odd numbers, prime numbers, factorization of odd numbers.

## 1. Introduction

The results of this paper are based on the following representation of odd numbers. Let  $\Pi$  be an odd number other than 1, and  $\left\lfloor \frac{\ln \Pi}{\ln 2} \right\rfloor$  is the integer part of  $\frac{\ln \Pi}{\ln 2} \in \mathbb{R}$ . Then  $\Pi$  can be uniquely written in the form

$$\Pi = 2^{\nu+1} + 2^\nu + \sum_{i=0}^{\nu-1} \beta_i 2^i,$$

where  $\nu \in \mathbb{N}$ ,  $\nu + 1 = \left\lfloor \frac{\ln \Pi}{\ln 2} \right\rfloor$ ,  $\beta_i = \pm 1, i = 0, 1, 2, \dots, \nu - 1$ .

(Theorem 1). It is easy to show that in such a representation, the powers of 2 cannot be replaced by powers of any other integer. The main difference between this result and the known arithmetic systems (binary, decimal etc.) is that the coefficients of the linear combination can take values  $-1$  or  $+1$ . This has the consequence that representation (1) has the largest number of terms, of all possible systems of representing odd numbers. This feature will allow us to define “the conjugate”, “the complementary”, “the L/R symmetry”, “the transpose” and “the K numbers” of an odd number.

The symmetry *L/R* (Definition 3) categorizes the odd numbers into four classes,  $Q$ ,  $V$ ,  $U$ , and  $D$  (Definition 5). The transpose (Definition 4) categorizes  $Q$ ,  $D$  and  $V$ ,  $U$  into “symmetric” and “asymmetric” numbers.

Combining the conjugate (Definition 1) with the transpose gives “the octets” of odd numbers (Definition 6). The octets of odd numbers are the most elegant object that emerges from Theorem 1. We study the internal structure of an octet and the relationship between two different octets.

In Section 8 we study sequences  $A(N) = 2^N + 1$  and  $B(N) = 2^N - 1$ . The factors of these sequences have a particular structure, which we present. If  $P$  is a prime factor of  $A(N) = 2^N + 1$  or  $B(N) = 2^N - 1$ , then  $P - 1$  has at least one common factor with  $N$  (Theorem 12).

In Section 9 we introduce “the K numbers” (Definition 9). These numbers are related to the structure of the odd numbers. One of them is the odd number  $K$  that appears in the *L/R* symmetry (refer to Proposition 3).

In Section 10 we make an application of the symmetries of the natural numbers. The combination of the  $K$  numbers with the octets of the odd numbers gives an algorithm for factoring the odd numbers.

## 2. Odd numbers as linear combinations of consecutive powers of 2

In this Section, we derive the representation mentioned in the Introduction.

**Theorem 1.** Let  $\Pi$  be an odd number other than 1, and  $\left\lfloor \frac{\ln \Pi}{\ln 2} \right\rfloor$  is the integer part of  $\frac{\ln \Pi}{\ln 2} \in \mathbb{R}$ . Then  $\Pi$  can be uniquely written in the form

$$\Pi = 2^{\nu+1} + 2^\nu + \sum_{i=0}^{\nu-1} \beta_i 2^i, \quad (1)$$

where  $\nu \in \mathbb{N}$ ,  $\nu+1 = \left\lfloor \frac{\ln \Pi}{\ln 2} \right\rfloor$ ,  $\beta_i = \pm 1, i = 0, 1, 2, \dots, \nu-1$ .

*Proof.* For binary intervals we follow the notation  $\Omega_\nu = [2^{\nu+1}, 2^{\nu+2}]$ ,  $\nu = -1, 0, 1, \dots$

If  $\Pi = 1$  we have  $\nu+1 = \left\lfloor \frac{\ln 1}{\ln 2} \right\rfloor = 0 \Rightarrow \nu = -1$  and

$$\Pi = 1 \in \Omega_{-1} = [2^0, 2^1] = [1, 2].$$

If  $\Pi = 3$  we have  $\nu+1 = \left\lfloor \frac{\ln 3}{\ln 2} \right\rfloor \Rightarrow \nu = 0$  and from Equation (1) we obtain

$$\Pi = 3 = 2^1 + 2^0 \in \Omega_0 = [2^1, 2^2] = [2, 2^2].$$

We now examine the case where  $\nu \in \mathbb{N}^* = \{1, 2, 3, \dots\}$ . For a particular value of  $\nu$ , the smallest value of the odd number (1) is

$$\Pi_{\min} = \Pi(\nu) = 2^{\nu+1} + 2^\nu - 2^{\nu-1} - 2^{\nu-2} - \dots - 2^1 - 1 = 2^{\nu+1} + 1. \quad (2)$$

The largest value of odd number (1) is

$$\Pi_{\max} = \Pi(\nu) = 2^{\nu+1} + 2^\nu + 2^{\nu-1} + 2^{\nu-2} + \dots + 2^1 + 1 = 2^{\nu+2} - 1. \quad (3)$$

From Equations (2) and (3) we get that for any odd  $\Pi = \Pi(\nu, \beta_i)$  in Equation (1), the following inequality holds,

$$\Pi_{\min} = 2^{\nu+1} + 1 \leq \Pi(\nu, \beta_i) \leq 2^{\nu+2} - 1 = \Pi_{\max}. \quad (4)$$

The number  $N(\Pi(\nu, \beta_i))$  of odd numbers in the closed interval  $[2^{\nu+1} + 1, 2^{\nu+2} - 1]$  is

$$N(\Pi(\nu, \beta_i)) = \frac{\Pi_{\max} - \Pi_{\min}}{2} + 1 = \frac{(2^{\nu+2} - 1) - (2^{\nu+1} + 1)}{2} + 1 = 2^\nu. \quad (5)$$

The integers  $\beta_i, i = 0, 1, 2, \dots, \nu-1$  in (1) can only take two values, namely  $\beta_i = \pm 1$ , and thus (1) gives exactly  $2^\nu = N(\Pi(\nu, \beta_i))$  odd numbers. Considering also Equation (5), we conclude that for every  $\nu \in \mathbb{N}^*$  Equation (1) gives all odd numbers in the interval  $\Omega_\nu$ .

From Inequality (4), we obtain

$$2^{\nu+1} + 1 \leq \Pi \leq 2^{\nu+2} - 1$$

so we have  $2^{\nu+1} < \Pi < 2^{\nu+2}$ . Thus

$$(\nu+1)\ln 2 < \ln \Pi < (\nu+2)\ln 2$$

from which we get

$$\frac{\ln \Pi}{\ln 2} - 1 < \nu + 1 < \frac{\ln \Pi}{\ln 2}$$

and finally

$$\nu + 1 = \left\lfloor \frac{\ln \Pi}{\ln 2} \right\rfloor. \quad (6)$$

We prove now that every odd number  $\Pi \neq 1$  can be uniquely written in the form of Equation (1). We write the odd  $\Pi$  as

$$\Pi = 2^{\nu+1} + 2^\nu + \sum_{i=0}^{\nu-1} \beta_i 2^i, \quad (7)$$

where  $\nu+1 = \left\lceil \frac{\ln \Pi}{\ln 2} \right\rceil$ ,  $\beta_i = \pm 1, i = 0, 1, 2, \dots, \nu-1$ , and

$$\Pi = 2^{\nu+1} + 2^\nu + \sum_{i=0}^{\nu-1} \gamma_i 2^i, \quad (8)$$

where  $\nu+1 = \left\lceil \frac{\ln \Pi}{\ln 2} \right\rceil$  and  $\gamma_i = \pm 1, i = 0, 1, 2, \dots, \nu-1$ .

From Equations (7) and (8) we get

$$(\beta_0 - \gamma_0) \cdot 2^0 + (\beta_1 - \gamma_1) \cdot 2^1 + (\beta_2 - \gamma_2) \cdot 2^2 + \dots + (\beta_{\nu-1} - \gamma_{\nu-1}) \cdot 2^{\nu-1} = 0, \quad (9)$$

where

$$\beta_i = \pm 1, i = 0, 1, 2, \dots, \nu-1, \quad i \in \{0, 1, 2, \dots, \nu-1\} \quad \text{and} \quad \gamma_i = \pm 1, i = 0, 1, 2, \dots, \nu-1, \quad i \in \{0, 1, 2, \dots, \nu-1\}.$$

If in Equation (9) there are  $i \in \{0, 1, 2, \dots, \nu-1\}$  such that  $\beta_i \neq \gamma_i$ , and let  $k$  is the smallest of them, then dividing by  $2^{k+1}$ , we get an odd number equal to an even number. So, it follows that  $\beta_i = \gamma_i \forall i = 0, 1, 2, \dots, \nu-1$ .

□

**Algorithm for representing an odd number in the form (1).** In order to write an odd number  $\Pi \neq 1, 3$  in the form of Equation (1) we initially define the  $\nu \in \mathbb{N}^*$  from Equation (6). Then, we calculate the sum  $2^{\nu+1} + 2^\nu$ .

If  $2^{\nu+1} + 2^\nu < \Pi$  we add  $2^{\nu-1}$ , whereas if  $2^{\nu+1} + 2^\nu > \Pi$  then we subtract it. By repeating the process exactly  $\nu$  times we write the odd number  $\Pi$  in the form of Equation (1). The number  $\nu$  of steps needed in order to write the odd number  $\Pi$  in the form of Equation (1) is extremely low compared to the magnitude of the odd number  $\Pi$ , as derived from Inequality (4).

**Example 1.** For the odd number  $\Pi = 23$  we obtain from Equation (6)

$$\nu+1 = \left\lceil \frac{\ln 23}{\ln 2} \right\rceil \Rightarrow \nu = 3.$$

Then, we have

$$2^{\nu+1} + 2^\nu = 2^4 + 2^3 = 24 > 23 \quad (\text{thus } 2^2 \text{ is subtracted})$$

$$2^4 + 2^3 - 2^2 = 20 < 23 \quad (\text{thus } 2^1 \text{ is added})$$

$$2^4 + 2^3 - 2^2 + 2^1 = 22 < 23 \quad (\text{thus } 2^0 = 1 \text{ is added})$$

$$2^4 + 2^3 - 2^2 + 2^1 + 1 = 23.$$

Fermat numbers  $F_s$  can be written directly in the form of Equation (1), since they are of the form  $\Pi_{\min}$ ,

$$F_s = 2^{2^s} + 1 = \Pi_{\min} (2^s - 1) = 2^{2^s} + 2^{2^s-1} - 2^{2^s-2} - 2^{2^s-3} - \dots - 2^1 - 1, \quad (10)$$

where  $s \in \mathbb{N}$ . Similarly, the Mersenne numbers  $M_p$  can be written directly in the form of Equation (1), since they are of the form  $\Pi_{\max}$ ,

$$M_p = 2^p - 1 = \Pi_{\max} (p - 2) = 2^{p-1} + 2^{p-2} + 2^{p-3} + \dots + 2^1 + 1, \quad (11)$$

where  $p$  is prime.

We now give the following definition.

**Definition 1.** Let  $\Pi$  be an odd number greater than 1, and consider the representation of  $\Pi$  as described in Theorem 1. Then the conjugate  $\Pi^*$  of  $\Pi$  is

$$\Pi^* = \Pi^*(\nu, \gamma_j) = 2^{\nu+1} + 2^\nu + \sum_{j=0}^{\nu-1} \gamma_j 2^j, \quad (12)$$

where  $\gamma_j = -\beta_j$ ,  $j = 0, 1, 2, \dots, \nu-1$ .

**Proposition 1.** *If  $\Pi$  is odd, then the following hold.*

1. *We have that*

$$(\Pi^*)^* = \Pi. \quad (13)$$

2. *We have that*

$$\Pi^* = 3 \cdot 2^{\nu+1} - \Pi. \quad (14)$$

3. *We have that  $\Pi$  is divisible by 3 if and only if  $\Pi^*$  is divisible by 3.*

4. *Two conjugate odd numbers cannot have common factors greater than 3.*

5. *Conjugates  $\Pi$  and  $\Pi^*$  are equidistant from the midpoint  $3 \cdot 2^\nu$  of the interval  $\Omega_\nu = [2^{\nu+1}, 2^{\nu+2}]$ .*

*Proof.* 1. This is an immediate consequence of Definition 1.

2. From Equations (1) and (12) we get

$$\Pi + \Pi^* = (2^{\nu+1} + 2^\nu) + (2^{\nu+1} + 2^\nu)$$

or equivalently

$$\Pi + \Pi^* = 3 \cdot 2^{\nu+1}.$$

3. If the odd  $\Pi$  is divisible by 3 then it is written in the form  $\Pi = 3x$ ,  $x = \text{odd}$ , and from Equation (14) we get

$$3x + \Pi^* = 3 \cdot 2^{\nu+1}, \text{ that is, } \Pi^* = 3(2^{\nu+1} - x). \text{ The converse can be proved similarly.}$$

4. Let  $\Pi = xy$ ,  $\Pi^* = xz$ ,  $x, y, z$  are odd numbers. Then, Equation (14) implies that  $x(y+z) = 3 \cdot 2^{\nu+1}$ , and consequently  $x = 3$ .

5. From equation (14) we obtain

$$\Pi - 3 \cdot 2^\nu = 3 \cdot 2^\nu - \Pi^*,$$

and thus

$$|\Pi - 3 \cdot 2^\nu| = |3 \cdot 2^\nu - \Pi^*|. \quad \square$$

Proposition 1 implies that 3 is the only odd number which is equal to its conjugate;  $3^* = 3 \cdot 2^{0+1} - 3 = 3$ . For the  $\Pi = 1$ , we define

$$1^* = 1. \quad (15)$$

It is easily proven that Theorem 1 is also valid for even numbers that are not powers of 2. In order to write an even number  $E$  that is not a power of 2 in the form of Equation (1), initially it is consecutively divided by 2 and it takes the form of equation

$$E = 2^l \cdot \Pi, \quad (16)$$

where  $\Pi$  odd number,  $\Pi \neq 1$ ,  $l \in \mathbb{N}^*$ . Then, we express  $\Pi$  as in Equation (1).

**Example 2.** By consecutively dividing the even number 368 by 2 we obtain

$$E = 368 = 2^4 \cdot 23.$$

Then, we write the odd number  $\Pi = 23$  in the form of Equation (1),

$$23 = 2^4 + 2^3 - 2^2 + 2^1 + 1$$

and we get

$$368 = 2^4 \cdot (2^4 + 2^3 - 2^2 + 2^1 + 1) = 2^8 + 2^7 - 2^6 + 2^5 + 2^4.$$

This equation gives the unique way in which the even number 368 can be written in the form of Equation (1). For even numbers the lowest power of two in Equation (1) is different from  $1 = 2^0$ .

The middle  $3 \cdot 2^\nu$  of the interval  $\Omega_\nu = [2^{\nu+1}, 2^{\nu+2}]$  is a center of symmetry of two conjugates numbers of the interval  $\Omega_\nu$ . Therefore, also for the even numbers  $E$  of  $\Omega_\nu$ , the conjugate  $E^*$  is defined, and the Equation  $E + E^* = 3 \cdot 2^{\nu+1}$  applies.

We now give the following definition.

**Definition 2.**

We define as *complementary* the odd numbers  $\Pi$  and  $\Pi'$  of the interval  $\Omega_\nu$ , for which  $\Pi' - \Pi = 2^\nu$  holds.

For complementary numbers the following holds.

**Proposition 2.**

1. We have that

$$(\Pi')' = \Pi.$$

2.  $\Pi$  and  $\Pi'$  have no common factor.

3. The complement and conjugate of an odd number  $\Pi$  are commutative,

$$(\Pi^*)' = (\Pi')^*.$$

*Proof.* 1. This is an immediate consequence of Definition 2.

2. Let  $\Pi = xy, \Pi' = xz$ ,  $x, y, z$  are odd numbers. From Definition 2 we get  $\Pi' - \Pi = 2^\nu$  or equivalently  $xz - xy = 2^\nu$  or equivalently  $x(z - y) = 2^\nu$ , which is impossible since the odd number  $x$  cannot be a factor of  $2^\nu$ .

3. We prove 3 of the proposition in the case where  $\Pi \in [2^{\nu+1}, 3 \cdot 2^\nu] \subseteq \Omega_\nu$ . The proof in the case where

$\Pi \in [3 \cdot 2^\nu, 2^{\nu+2}] \subseteq \Omega_\nu$  is similar. We have  $\Pi' = \Pi + 2^\nu$  and

$$(\Pi')^* = 3 \cdot 2^{\nu+1} - \Pi' = 3 \cdot 2^{\nu+1} - (\Pi + 2^\nu) = 3 \cdot 2^{\nu+1} - \Pi - 2^\nu = 2^{\nu+2} - \Pi.$$

Since  $\Pi \in [2^{\nu+1}, 3 \cdot 2^\nu] \subseteq \Omega_\nu$ , we have  $\Pi^* \in [3 \cdot 2^\nu, 2^{\nu+2}] \subseteq \Omega_\nu$ . So we get

$$(\Pi^*)' = \Pi^* - 2^\nu = 3 \cdot 2^{\nu+1} - \Pi - 2^\nu = 2^{\nu+2} - \Pi.$$

Therefore we have  $(\Pi^*)' = (\Pi')^*$ .

□

### 3. The L/R symmetry

We now give the following definition.

**Definition 3.**

1. The odd number  $\Pi$  has *Left-symmetry*  $L$  when there exists an index  $L$  such that

$$\beta_{L-1} = +1, \beta_{L-2} = \beta_{L-3} = \beta_{L-4} = \dots = \beta_2 = \beta_1 = \beta_0 = -1, \tag{17}$$

where  $L \in \{2, 3, 4, \dots, \nu - 1\}$ .

2. The odd number  $\Pi$  has *Right-symmetry*  $R$  when there exists an index  $R$  such that

$$\beta_{R-1} = -1, \beta_{R-2} = \beta_{R-3} = \beta_{R-4} = \dots = \beta_2 = \beta_1 = \beta_0 = +1, \tag{18}$$

where  $R \in \{2, 3, 4, \dots, \nu - 1\}$ .

Sometimes, we will use the notation  $L(\Pi)$  and  $R(\Pi)$  for  $L$  and  $R$ , respectively, above.

**Example 3.** The prime number

$$Q = 568630647535356955169033410940867804839360742060818433$$

is a factor of  $F_{12} = 2^{4096} + 1$ . From the Equation (6) we have  $\nu + 1 = 178$ , and then from Equation (1) we have

$$\begin{aligned}
Q = & 2^{178} + 2^{177} - 2^{176} + 2^{175} + 2^{174} + 2^{173} + 2^{172} - 2^{171} + 2^{170} + 2^{169} + 2^{168} + 2^{167} + 2^{166} \\
& + 2^{165} - 2^{164} + 2^{163} - 2^{162} - 2^{161} - 2^{160} - 2^{159} + 2^{158} + 2^{157} + 2^{156} - 2^{155} - 2^{154} - 2^{153} - 2^{152} \\
& - 2^{151} + 2^{150} - 2^{149} + 2^{148} - 2^{147} - 2^{146} + 2^{145} - 2^{144} + 2^{143} - 2^{142} - 2^{141} - 2^{140} + 2^{139} + 2^{138} \\
& - 2^{137} - 2^{136} + 2^{135} - 2^{134} - 2^{133} + 2^{132} - 2^{131} + 2^{130} - 2^{129} + 2^{128} - 2^{127} + 2^{126} - 2^{125} - 2^{124} \\
& - 2^{123} - 2^{122} - 2^{121} + 2^{120} - 2^{119} + 2^{118} - 2^{117} + 2^{116} - 2^{115} + 2^{114} - 2^{113} - 2^{112} - 2^{111} - 2^{110} \\
& - 2^{109} - 2^{108} + 2^{107} - 2^{106} + 2^{105} - 2^{104} + 2^{103} - 2^{102} + 2^{101} - 2^{100} + 2^{99} + 2^{98} - 2^{97} + 2^{96} - 2^{95} \\
& - 2^{94} + 2^{93} - 2^{92} + 2^{91} + 2^{90} - 2^{89} + 2^{88} - 2^{87} + 2^{86} + 2^{85} + 2^{84} - 2^{83} + 2^{82} - 2^{81} + 2^{80} + 2^{79} \\
& - 2^{78} - 2^{77} - 2^{76} - 2^{75} + 2^{74} + 2^{73} - 2^{72} - 2^{71} - 2^{70} + 2^{69} + 2^{68} + 2^{67} + 2^{66} + 2^{65} + 2^{64} - 2^{63} \\
& - 2^{62} + 2^{61} - 2^{60} - 2^{59} - 2^{58} - 2^{57} - 2^{56} + 2^{55} - 2^{54} - 2^{53} - 2^{52} - 2^{51} - 2^{50} - 2^{49} + 2^{48} + 2^{47} \\
& - 2^{46} + 2^{45} + 2^{44} + 2^{43} + 2^{42} - 2^{41} - 2^{40} + 2^{39} - 2^{38} - 2^{37} - 2^{36} + 2^{35} - 2^{34} - 2^{33} + 2^{32} + 2^{31} \\
& - 2^{30} + 2^{29} + 2^{28} + 2^{27} + 2^{26} + 2^{25} + 2^{24} - 2^{23} + 2^{22} + 2^{21} + 2^{20} - 2^{19} - 2^{18} - 2^{17} - 2^{16} + 2^{15} \\
& + 2^{14} - 2^{13} - 2^{12} - 2^{11} - 2^{10} - 2^9 - 2^8 - 2^7 - 2^6 - 2^5 - 2^4 - 2^3 - 2^2 - 2^1 - 1
\end{aligned}$$

So the factor 568630647535356955169033410940867804839360742060818433 of  $F_{12}$  has Left-symmetry  $L(568630647535356955169033410940867804839360742060818433)=15$ .

From the Euclidean division identity it follows that every odd number  $\Pi$  is written in one of the following forms,

$$Q = Q_m = 8m + 1,$$

$$V = V_m = 8m + 3,$$

$$U = U_m = 8m + 5,$$

$$D = D_m = 8m + 7,$$

where  $m \in \mathbb{N}$ .

From these Equations we obtain the following proposition.

**Proposition 3.**

1.  $Q = Q_0 = 1$  or  $Q = 2^L K + 1$ , where  $K$  is an odd number and  $L \geq 3$ .
2.  $V = V_0 = 3$  or  $V = 2^R K + 3$ , where  $K$  is an odd number and  $R \geq 3$ .
3.  $U = U_0 = 5$  or  $U = 2^L K + 5$ , where  $K$  is an odd number and  $L \geq 3$ .
4.  $D = D_0 = 7$  or  $D = 2^R K + 7$ , where  $K$  is an odd number and  $R \geq 3$ .

*Proof.* We prove 1, 2, 3 and 4 are proved similarly. We have  $Q = 8m + 1$ . If  $m = 0$  we have  $Q = 1$ . If  $m \neq 0$  is an odd number,  $m = K$ , we have  $Q = 2^3 \cdot K + 1$ . If  $m \neq 0$  is an even number,  $M = 2^n \cdot K$ ,  $n = 1, 2, 3, \dots$ , we have  $Q = 2^{3+n} \cdot K + 1 = 2^L \cdot K + 1$  where  $L > 3$ . □

We give two examples.

**Example 4.** For odd number 18303 we have

$$18303 + 1 = 2^7 \cdot 143.$$

Therefore  $R(18303) = 7$ . Indeed, from Equation (6) we get  $\nu = 13$  and from Equation (1) we obtain

$$18303 = 2^{14} + 2^{13} - 2^{12} - 2^{11} - 2^{10} + 2^9 + 2^8 + 2^7 - 2^6 + 2^5 + 2^4 + 2^3 + 2^2 + 2^1 + 1.$$

**Example 5.** For the number C1133 which is composite factor of  $F_{12}$  with 1133 digits, we have

$$C1133 - 1 = 2^{14} \cdot K.$$

Therefore,  $L(C1133) = 14$ .

We now prove the following.

**Proposition 4.**

The numbers  $Q, V, U, D$  are written in the form

$$Q = 4M - 3,$$

$$V = 4M - 1,$$

$$U = 4M + 1,$$

$$D = 4M + 3,$$

where  $M = 1, 3, 5, \dots$

*Proof.* We prove the first Equation. The other three are proved similarly. We have

$$Q = 2^L \cdot K + 1, L \geq 3, K = 0, 1, 3, 5, \dots$$

or equivalently

$$Q = 2^L \cdot K + 4 - 3$$

or equivalently

$$Q = 4(2^{L-2} \cdot K + 1) - 3$$

or equivalently

$$Q = 4M - 3,$$

where  $M = 2^{L-2} \cdot K + 1 = 1, 3, 5, \dots$

□

From Definition 1 and Equations (21), (22) we obtain the following.

**Corollary 1.**

The conjugate of  $Q = 2^L \cdot K + 1$  is  $Q^* = 2^L \cdot K^* - 1$ , and vice versa.

It is easy to prove the following.

**Proposition 5.**

1.  $Q_1 Q_2 = Q$ .
2.  $D_1 D_2 = Q$ .
3.  $Q_1 D_1 = D$ .
4.  $L(Q_1) < L(Q_2) \Rightarrow L(Q_1 Q_2) = L(Q_1)$ .
5.  $L(Q) < R(D) \Rightarrow R(QD) = L(Q)$ .
6.  $R(D) < L(Q) \Rightarrow R(QD) = R(D)$ .
7.  $R(D_1) < R(D_2) \Rightarrow L(D_1 D_2) = R(D_1)$ .
8.  $Symmetry(\Pi_1) = Symmetry(\Pi_2) \Rightarrow Symmetry(\Pi_1 \Pi_2) > Symmetry(\Pi_1) = Symmetry(\Pi_2)$ .

We give two examples:

**Example 6.**  $L(641)=7 < L(114689)=14 \Rightarrow L(641 \times 114689)=7$ .

**Example 7.**  $R(607)=5 < R(16633)=6 \Rightarrow L(607 \times 16633)=5$ .

A consequence of Proposition 5 is the following.

**Corollary 2. 1.** Composite numbers of the form  $A = 2^{v+1} + 1$  are written as

$$A = 2^{v+1} + 1 = (2^x \cdot K_1 + 1)(2^x \cdot K_2 + 1), \quad x \geq 3,$$

where  $x = 1, 2, 3, \dots$  and  $K_1, K_2$  are odd numbers.

2. Composite numbers of the form  $B = 2^{v+1} - 1$  are written as

$$B = 2^{v+1} - 1 = (2^x \cdot K_1 + 1)(2^x \cdot K_2 - 1),$$

where  $x = 4, 5, 6, \dots$  and  $K_1, K_2$  are odd numbers.

From Definitions 1 and 2 we obtain the following.

**Corollary 3.** In every conjugate pair  $(\Pi, \Pi^*)$ , one number has left symmetry, and the other has right.

We now prove the following.

**Proposition 6.**

1. The squares of odd numbers are of the form  $Q$ .
2. The product  $\Pi Q$  of an odd number  $\Pi$  by an odd number of the form  $Q$  has the same form and the same symmetry value  $L/R$  as  $\Pi$ .
3. The even powers of an odd number  $\Pi$  are of the form  $Q$ .
4. The odd powers of an odd number  $\Pi$  have the same form and the same symmetry value  $L/R$  as  $\Pi$ .

*Proof.* 1. We prove that  $V^2$  is of form  $Q$ . The proof for  $Q^2$ ,  $U^2$  and  $D^2$  is similar. We have

$$V^2 = (8m+3)^2 = 8(8m^2 + 3m+1) + 1 = 8m' + 1.$$

2. We do the proof for the product  $VQ$ . The proof for the other three products is similar. We have

$$VQ = (8m+3)(8n+1) = 64mn + 8m + 24n + 3 = 8(8mn + m + 3n) + 3 = 8m' + 3.$$

3, 4. 3 and 4 of the proposition result from the combination of 1 and 2.

□

#### 4. Transpose of odd number. Categorization of odd numbers

We now give the following definition.

**Definition 4.** 1. We write the numbers  $\Delta = D$  or  $\Delta = V$  in the form of Equation (1),

$$\Delta = 2^{\nu+1} + 2^\nu + \beta_{\nu-1}2^{\nu-1} + \beta_{\nu-2}2^{\nu-2} + \dots + \beta_12^1 + 1, \quad (19)$$

where  $\nu+1 = \left\lceil \frac{\ln \Delta}{\ln 2} \right\rceil$ . We define the transpose  $T(\Delta)$  of  $\Delta$  as

$$T(\Delta) = \left( \frac{1}{2^{\nu+1}} + \frac{1}{2^\nu} + \frac{\beta_{\nu-1}}{2^{\nu-1}} + \frac{\beta_{\nu-2}}{2^{\nu-2}} + \dots + \frac{\beta_1}{2^1} + 1 \right) \cdot 2^{\nu+1} = 2^{\nu+1} + 3 + \sum_{k=1}^{\nu-1} \beta_k \cdot 2^{\nu+1-k}. \quad (20)$$

2. We write the numbers  $A = Q$  or  $A = U$  in the form of Equation (1),

$$A = 2^{\nu+1} + 2^\nu + \beta_{\nu-1}2^{\nu-1} + \beta_{\nu-2}2^{\nu-2} + \dots + \beta_12^1 - 1, \quad (21)$$

where  $\nu+1 = \left\lceil \frac{\ln A}{\ln 2} \right\rceil$ . We define the transpose  $T(A)$  of  $A$  as

$$T(A) = - \left( \frac{1}{2^{\nu+1}} + \frac{1}{2^\nu} + \frac{\beta_{\nu-1}}{2^{\nu-1}} + \frac{\beta_{\nu-2}}{2^{\nu-2}} + \dots + \frac{\beta_1}{2^1} - 1 \right) \cdot 2^{\nu+1} = 2^{\nu+1} - 3 - \sum_{k=1}^{\nu-1} \beta_k \cdot 2^{\nu+1-k}. \quad (22)$$

3. We set

$$T(1) = 1. \quad (23)$$

4. From Equations (20), (22), (23) we get the general equation

$$T(\Pi) = 2^{\nu+1} + \beta_0 \cdot \left( 3 + \sum_{k=1}^{\nu-1} \beta_k \cdot 2^{\nu+1-k} \right), \quad (24)$$

where  $\nu+1 = \left\lceil \frac{\ln \Pi}{\ln 2} \right\rceil$ .

**Algorithm for the calculation of the transpose.** Let  $\Pi$  be an odd number. We first calculate  $\nu+1 = \left\lceil \frac{\ln \Pi}{\ln 2} \right\rceil$  from

Equation (6). Next, applying the algorithm described of Example 1, we write  $\Pi$  in the form of Equation (1), and we calculate  $\beta_i = \pm 1, i = 1, 2, 3, \dots, \nu-1$  and the transpose  $T(\Pi)$  of  $\Pi$  from Equation (24).

We now prove a series of results regarding the transpose of an odd number.

**Theorem 2.**

1. It holds that

$$T(\Pi) = 1 \text{ if and only if } \Pi = 2^\nu - 3, \nu \geq 2, \nu \in \mathbb{N}. \quad (25)$$

2. It holds that

$$T(D) = D \left. \vphantom{T(D) = D} \right\} \Leftrightarrow \left\{ \begin{array}{l} \beta_1 = 1 \\ \beta_{\nu-k} = \beta_{k+1} \\ k = 1, 2, 3, \dots, \frac{\nu-2}{2}, \nu = \text{even}, \\ k = 1, 2, 3, \dots, \frac{\nu-1}{2}, \nu = \text{odd} \end{array} \right. \quad (26)$$

$$T(Q) = Q \left. \vphantom{T(Q) = Q} \right\} \Leftrightarrow \left\{ \begin{array}{l} \beta_1 = -1 \\ \beta_{\nu-k} = -\beta_{k+1} \\ k = 1, 2, 3, \dots, \frac{\nu-2}{2}, \nu = \text{even}, \\ k = 1, 2, 3, \dots, \frac{\nu-1}{2}, \nu = \text{odd} \end{array} \right. \quad (27)$$

*Proof.* 1. For  $\Pi = 2^\nu - 3$  we get

$$\begin{aligned} \Pi &= 2^\nu - 3 = (2^\nu - 1) - 2 = (2^{\nu-1} + 2^{\nu-2} + 2^{\nu-2} + \dots + 2^1 + 1) - 2 \\ &= 2^{\nu-1} + 2^{\nu-2} + 2^{\nu-2} + \dots + 2^1 - 1 \end{aligned}$$

that is,  $\Pi$  has left symmetry, and thus Equation (22) implies  $T(\Pi) = 1$ . Now, let  $T(\Pi) = 1$ . The odd  $\Pi$  has either left or right symmetry. We only consider the former case, as the latter is similar. Then

$$\Pi = 2^{n+1} + 2^n + \beta_{n-1}2^{n-1} + \beta_{n-2}2^{n-2} + \dots + \beta_22^2 + \beta_12^1 - 1, \quad (28)$$

where  $n+1 = \left\lceil \frac{\ln \Pi}{\ln 2} \right\rceil$ . Thus, from Equation (22), we get

$$T(\Pi) = -1 - 2^1 - \beta_{n-1}2^2 - \dots - \beta_22^n + 2^{n+1}.$$

Thus, we have the following sequence of equivalent equations

$$T(\Pi) = 1$$

$$-1 - 2^1 - \beta_{n-1}2^2 - \dots - \beta_22^n + 2^{n+1} = 1$$

$$-1 - 2^1 - \beta_{n-1}2^2 - \dots - \beta_22^n + 2^{n+1} + 2^{n+2} = 2^{n+2} + 1$$

$$-1 - 2^1 - \beta_{n-1}2^2 - \dots - \beta_22^n + 2^{n+1} + 2^{n+2} = 2^{n+2} + 2^n - 2^{n-1} - 2^{n-2} - \dots - 2^2 - 2^1 - 1$$

As the representation in Equation (1) is unique, we have

$$\beta_2 = \beta_3 = \beta_3 = \dots = \beta_{n-1} = +1.$$

Therefore, (28) implies

$$\Pi = 2^{n+1} + 2^n + 2^{n-1} \dots + 2^1 - 1 = 2(2^n + 2^{n-1} + 2^{n-2} + \dots + 2^1 + 1) - 1$$

$$= 2(2^{n+1} - 1) - 1 = 2^{n+2} - 3$$

Setting  $n+2 = \nu$  we obtain  $\Pi = 2^\nu - 3$ .

2. This is proved similarly. We write  $D$  in the form of Equation (1), that is,

$$D = 2^{\nu+1} + 2^\nu + \beta_{\nu-1}2^{\nu-1} + \beta_{\nu-2}2^{\nu-2} + \dots + \beta_1 2^1 + 1, \quad (29)$$

where  $\nu + 1 = \left\lceil \frac{\ln D}{\ln 2} \right\rceil$ . Equations (29) and (20) imply

$$T(D) = 1 + 2 + \beta_{\nu-1}2^2 + \beta_{\nu-2}2^3 + \dots + \beta_2 2^\nu + 2^{\nu+1}. \quad (30)$$

Then  $T(D) = D$  if and only if

$$1 + 2 + \beta_{\nu-1}2^2 + \beta_{\nu-2}2^3 + \dots + \beta_2 2^\nu + 2^{\nu+1} = 2^{\nu+1} + 2^\nu + \beta_{\nu-1}2^{\nu-1} + \beta_{\nu-2}2^{\nu-2} + \dots + \beta_2 2^2 + \beta_1 2^1 + 1.$$

As the representation in (1) is unique, the proof is complete.  $\square$

**Theorem 3.**

1. If the odd number  $D$  has right symmetry, then

$$T(D) - T(D^*) = 6. \quad (31)$$

2. If the odd number  $Q$  has left symmetry, then

$$T(Q) - T(Q^*) = -6. \quad (32)$$

*Proof.* We only prove (31), as the proof of (32) is similar. From Equation (29), we get,

$$D^* = 2^{\nu+1} + 2^\nu - \beta_{\nu-1}2^{\nu-1} - \beta_{\nu-2}2^{\nu-2} - \dots - \beta_1 2^1 - 1 = Q. \quad (33)$$

Equation (22) implies

$$T(D^*) = -1 - 2 + \beta_{\nu-1}2^2 + \beta_{\nu-2}2^3 + \dots + \beta_1 2^\nu + 2^{\nu+1}. \quad (34)$$

Finally, from (30) and (34), we obtain  $T(D) - T(D^*) = 6$ .  $\square$

**Theorem 4.**

For every odd  $\Pi$ ,  $\nu + 1 = \left\lceil \frac{\ln \Pi}{\ln 2} \right\rceil$ ,  $\Pi \in \Omega_\nu = [2^{\nu+1}, 2^{\nu+2}]$ , then

$$T(\Pi) < 2^{\nu+2}. \quad (35)$$

*Proof.* Without loss of generality, we may assume that  $\Pi$  has right symmetry. From Equation (30), and taking into account that  $\beta_i = \pm 1, i = 0, 1, 2, \dots, \nu - 1$ , we obtain

$$\begin{aligned} T(D) &= 1 + 2 + \beta_{\nu-1}2^2 + \beta_{\nu-2}2^3 + \dots + \beta_2 2^\nu + 2^{\nu+1} \leq 1 + 2 + 2^2 + \dots + 2^\nu + 2^{\nu+1} = 2^{\nu+2} - 1 \\ T(D) &\leq 2^{\nu+2} - 1 < 2^{\nu+2} \end{aligned} \quad \square$$

This result implies that if an odd  $\Pi$  belongs to the interval  $\Omega_\nu = [2^{\nu+1}, 2^{\nu+2}]$ , its transpose  $T(\Pi)$  can be found in intervals  $\Omega_n, n \leq \nu$ .

**Theorem 5.**

If  $D, Q$  belongs to the interval  $\Omega_\nu = [2^{\nu+1}, 2^{\nu+2}]$ ,  $\nu + 1 = \left\lceil \frac{\ln D}{\ln 2} \right\rceil = \left\lceil \frac{\ln Q}{\ln 2} \right\rceil$ ,  $\nu = 4, 5, 6, \dots$ , then

$$T(D-2) + T(D) = T(Q) + T(Q+2) = 2^{\nu+2}. \quad (36)$$

*Proof.* The smallest odd number  $D$  with right symmetry in  $\Omega_\nu = [2^{\nu+1}, 2^{\nu+2}]$  is  $D_{\min} = 2^{\nu+1} + 3$ . Thus,  $D \in \Omega_\nu$  if and only if  $(D-2) \in \Omega_\nu$  ( $D > 3$ ). The largest odd number with left symmetry in the interval  $\Omega_\nu$  is  $Q_{\min} = 2^{\nu+1} + 1$ . Thus, the following  $Q \in \Omega_\nu$  if and only if  $(Q+2) \in \Omega_\nu$  ( $Q > 3$ ). We do the proof of the equation  $T(Q) + T(Q+2) = 2^{\nu+2}$ . The proof of the equation  $T(D-2) + T(D) = 2^{\nu+2}$  is similar. From Definition 3 we have

$$Q = 2^{\nu+1} + 2^\nu + \beta_{\nu-1}2^{\nu-1} + \beta_{\nu-2}2^{\nu-2} + \beta_{\nu-3}2^{\nu-3} + \dots + \beta_2 2^2 - 2 - 1$$

and

$$Q + 2 = 2^{\nu+1} + 2^\nu + \beta_{\nu-1}2^{\nu-1} + \beta_{\nu-2}2^{\nu-2} + \beta_{\nu-3}2^{\nu-3} + \dots + \beta_22^2 - 2 + 1.$$

From these Equations and (24) we get

$$T(Q) + T(Q + 2) = 2^{\nu+2}.$$

□

**Theorem 6.**

If  $D, Q$  belongs to the interval  $\Omega_\nu = [2^{\nu+1}, 2^{\nu+2}]$ ,  $\nu + 1 = \left\lceil \frac{\ln D}{\ln 2} \right\rceil = \left\lceil \frac{\ln Q}{\ln 2} \right\rceil$ ,  $\nu = 4, 5, 6, \dots$ , then

$$T(D) - T(D - 4) = T(Q) - T(Q + 4) = 2^{\nu+1}. \quad (37)$$

*Proof.* We do the proof of the equation  $T(Q) - T(Q + 4) = 2^{\nu+1}$ .

The proof of the equation  $T(D) - T(D - 4) = 2^{\nu+1}$  is similar. From Definition 3 we have

$$Q = 2^{\nu+1} + 2^\nu + \beta_{\nu-1}2^{\nu-1} + \beta_{\nu-2}2^{\nu-2} + \beta_{\nu-3}2^{\nu-3} + \dots + \beta_22^2 - 2 - 1$$

and

$$Q + 4 = 2^{\nu+1} + 2^\nu + \beta_{\nu-1}2^{\nu-1} + \beta_{\nu-2}2^{\nu-2} + \beta_{\nu-3}2^{\nu-3} + \dots + \beta_22^2 + 2 - 1.$$

From these Equations and (24) we get

$$T(Q) - T(Q + 4) = 2^{\nu+1}.$$

□

Expunging  $T(D)$  and  $T(Q)$  from Equations (36) and (37) we obtain the following.

**Corollary 4.**

If  $D, Q$  belongs to the interval  $\Omega_\nu = [2^{\nu+1}, 2^{\nu+2}]$ ,  $\nu + 1 = \left\lceil \frac{\ln D}{\ln 2} \right\rceil = \left\lceil \frac{\ln Q}{\ln 2} \right\rceil$ ,  $\nu = 4, 5, 6, \dots$ , then

$$T(D - 2) + T(D - 4) = T(Q + 2) + T(Q + 4) = 2^{\nu+1}. \quad (38)$$

**Theorem 7.**

For every odd  $\Pi$ , then

$$T(2^n \cdot \Pi) = T(\Pi),$$

where  $n \in \mathbb{N}$ .

*Proof.* We prove the Theorem for odd D ones with left symmetry. The proof for odd Q ones with right symmetry is similar. We have

$$D = 2^{\nu+1} + 2^\nu + \beta_{\nu-1}2^{\nu-1} + \beta_{\nu-2}2^{\nu-2} + \dots + \beta_12^1 + 1,$$

$$\text{where } \nu + 1 = \left\lceil \frac{\ln D}{\ln 2} \right\rceil.$$

From this Equation and (20) we get the following equivalent equations,

$$2^n D = 2^{n+\nu+1} + 2^{n+\nu} + \beta_{\nu-1}2^{n+\nu-1} + \beta_{\nu-2}2^{n+\nu-2} + \dots + \beta_12^{n+1} + 2^n$$

$$T(2^n D) = \left( \frac{1}{2^{n+\nu+1}} + \frac{1}{2^{n+\nu}} + \frac{\beta_{\nu-1}}{2^{n+\nu-1}} + \frac{\beta_{\nu-2}}{2^{n+\nu-2}} + \dots + \frac{\beta_1}{2^{n+1}} + \frac{1}{2^n} \right) \cdot 2^{n+\nu+1}$$

$$= \left( \frac{1}{2^{\nu+1}} + \frac{1}{2^\nu} + \frac{\beta_{\nu-1}}{2^{\nu-1}} + \frac{\beta_{\nu-2}}{2^{\nu-2}} + \dots + \frac{\beta_1}{2^1} + \frac{1}{2^0} \right) \cdot 2^{\nu+1} = T(D)$$

□

We now give the following definition.

**Definition 5.** Categorizing odd numbers. Let  $\Pi$  be an odd number.

1. We define as *symmetric* every odd  $\Pi$  for which

$$T(T(\Pi)) = \Pi. \quad (39)$$

2. We define as *asymmetric* every odd  $\Pi$  for which

$$T(T(\Pi)) \neq \Pi. \quad (40)$$

From Equations (20) and (22) we get the following.

**Corollary 5.**

1.  $D$  and  $Q$  are symmetric numbers.

2.  $V$  and  $U$  are asymmetric numbers.

From Equation (24) and Definition 5 we obtain the following.

**Corollary 6.**

1. The odd number  $T(Q)$  is of the form  $Q$ .

2. The odd number  $T(D)$  is of the form  $D$ .

3. The odd number  $T(U)$  is of the form  $Q$ .

4. The odd number  $T(V)$  is of the form  $D$ .

5.  $T(\Pi) = \Pi$  if and only if  $\Pi$  is a symmetric number or  $\Pi = 3$ .

**Theorem 8.**

If an asymmetric odd number belongs to the interval  $\Omega_\nu$ , then its transpose belongs to the interval  $\Omega_\mu$  with  $\mu < \nu$ .

*Proof.* This is a direct consequence of Equation (24) and Proposition 3. □

Equation (31) has been proved for all odd numbers with right symmetry. Equation (32) has been proved for all odd numbers with left symmetry. Thus from Theorem 3 we obtain the following.

**Corollary 7.**

1. For an asymmetric number of the form  $V$ ,

$$T(V) - T(V^*) = 6.$$

2. For an asymmetric number of the form  $U$ ,

$$T(U) - T(U^*) = -6.$$

From Definition 4 we obtain the following.

**Corollary 8.**

1. Let  $a$  be an odd number belongs to the interval  $\Omega_\nu$ ,

$$Q = 2^L \cdot K + 1, \quad L \geq 4,$$

$K$  is odd, then  $T(Q+4)$  belongs to the interval  $\Omega_{\nu-1}$  and  $T(Q+2)$  belongs to the interval  $\Omega_{\nu-(L-2)}$ .

2. Let  $a$  be an odd number belongs to the interval  $\Omega_\nu$ ,

$$D = 2^R \cdot K - 1, \quad R \geq 4,$$

$K$  is odd, then  $T(D-4)$  belongs to the interval  $\Omega_{\nu-(R-2)}$  and  $T(D-2)$  belongs to the interval  $\Omega_{\nu-1}$ .

From Proposition 3 we obtain the following.

**Corollary 9.**

1. The product  $Q \cdot U$  is of the form  $U$ .

2. The product  $Q \cdot V$  is of the form  $V$ .

3. The product  $D \cdot U$  is of the form  $V$ .

4. The product  $D \cdot V$  is of the form  $U$ .

5. The product  $U \cdot V$  is of the form  $D$ .

6. The product  $V_1 \cdot V_2$  is of the form  $Q$ .

7. The product  $U_1 \cdot U_2$ , is of the form  $Q$ .

From Proposition 5 and Corollary 9 we obtain the following.

**Corollary 10.**

A composite odd number has one of the following ten forms.

1.  $Q = Q_1 Q_2$ .

2.  $Q = D_1 D_2$ .

3.  $Q = V_1 V_2$ .

4.  $Q = U_1 U_2$ .

5.  $V = Q_1 V_1$ .

6.  $V = D_1 U_1$ .

7.  $U = Q_1 U_1$ .

8.  $U = D_1 V_1$ .

9.  $D = Q_1 D_1$ .

10.  $D = U_1 V_1$ .

From Proposition 5 and Corollaries 5 and 9 we obtain the following.

**Corollary 11.**

For the odd numbers  $Q, V, U, D$  the following holds.

1. The product  $Q \cdot T(Q)$  is of the form  $Q$ .

2. The product  $V \cdot T(V)$  is of the form  $U$ .

3. The product  $U \cdot T(U)$  is of the form  $U$ .

4. The product  $D \cdot T(D)$  is of the form  $Q$ .

From Definition 2 we obtain the following.

**Corollary 12.**

If the complementary numbers  $\Pi$  and  $\Pi'$  belong to the interval  $\Omega_\nu$  with  $\nu \geq 3$ , then they are of the same form  $Q, V, U$  or  $D$ .

We now prove the following.

**Proposition 7.**

A. 1. If  $Q \in \Omega_\nu$  is a symmetric number with left symmetry, then

$$T(Q) = T(3 \cdot 2^{\nu+1} - Q) - 6.$$

2. If  $U \in \Omega_\nu$  is an asymmetric number with left symmetry, then

$$T(U) = T(3 \cdot 2^{\nu+1} - U) - 6.$$

B. 1. If  $D \in \Omega_\nu$  is a symmetric number with right symmetry, then

$$T(D) = T(3 \cdot 2^{\nu+1} - D) + 6.$$

2. If  $V \in \Omega_\nu$  is an asymmetric number with right symmetry, then

$$T(V) = T(3 \cdot 2^{\nu+1} - V) + 6.$$

*Proof.* The proposition is a consequence of Theorem 3 and Corollary 7. We prove A.1. A.2 and B.1, B.2 are proved similarly. From Equation (32) we get  $T(Q^*) - T(Q) = 6$  and from Equation (14) we get

$$T(3 \cdot 2^{\nu+1} - Q) - T(Q) = 6$$

or equivalently

$$T(Q) = T(3 \cdot 2^{\nu+1} - Q) - 6. \quad \square$$

**Theorem 9.**

1. For numbers  $\Delta = 2^{\nu+1} + 2^\nu + \beta_{\nu-1}2^{\nu-1} + \beta_{\nu-2}2^{\nu-2} + \beta_{\nu-3}2^{\nu-3} + \dots + \beta_22^2 + \beta_12^1 + 1$  we have

$$T(\Delta) - T(\Delta - \beta_k 2^{k+1}) = 2^{\nu+2-k} \beta_k, \quad (41)$$

where  $k = 1, 2, 3, \dots, \nu - 1$ .

2. For numbers  $A = 2^{\nu+1} + 2^\nu + \beta_{\nu-1}2^{\nu-1} + \beta_{\nu-2}2^{\nu-2} + \beta_{\nu-3}2^{\nu-3} + \dots + \beta_22^2 + \beta_12^1 - 1$  we have

$$T(A - \beta_k 2^{k+1}) - T(A) = 2^{\nu+2-k} \beta_k, \quad (42)$$

where  $k = 1, 2, 3, \dots, \nu - 1$ .

*Proof.* We prove Equation (41). (42) is proved similarly. We have

$$\Delta = 2^{\nu+1} + 2^\nu + \beta_{\nu-1}2^{\nu-1} + \beta_{\nu-2}2^{\nu-2} + \beta_{\nu-3}2^{\nu-3} + \dots + \beta_k 2^k + \dots + \beta_2 2^2 + \beta_1 2^1 + 1$$

and

$$\Delta - \beta_k 2^{k+1} = 2^{\nu+1} + 2^\nu + \beta_{\nu-1}2^{\nu-1} + \beta_{\nu-2}2^{\nu-2} + \beta_{\nu-3}2^{\nu-3} + \dots - \beta_k 2^k + \dots + \beta_2 2^2 + \beta_1 2^1 + 1.$$

From these Equations we get

$$T(\Delta) = 2^{\nu+1} \left( 1 + \frac{1}{2} + \frac{\beta_{\nu-1}}{2^{\nu-1}} + \frac{\beta_{\nu-2}}{2^{\nu-2}} + \frac{\beta_{\nu-3}}{2^{\nu-3}} + \dots + \frac{\beta_k}{2^k} + \dots + \frac{\beta_2}{2^2} + \frac{\beta_1}{2^1} + 1 \right),$$

$$T(\Delta - \beta_k 2^{k+1}) = 2^{\nu+1} \left( 1 + \frac{1}{2} + \frac{\beta_{\nu-1}}{2^{\nu-1}} + \frac{\beta_{\nu-2}}{2^{\nu-2}} + \frac{\beta_{\nu-3}}{2^{\nu-3}} + \dots - \frac{\beta_k}{2^k} + \dots + \frac{\beta_2}{2^2} + \frac{\beta_1}{2^1} + 1 \right)$$

$$\text{and finally we get } T(\Delta) - T(\Delta - \beta_k 2^{k+1}) = 2^{\nu+2-k} \beta_k.$$

□

**5. Octet of odd numbers**

We now give the following definitions.

**Definition 6.** We define as *the octet*  $\Phi$  of odd number  $\Pi$  the non ordered octet

$$\Phi = \left( \Pi, T(\Pi), (T(\Pi))^*, T\left((T(\Pi))^*\right), \Pi^*, T(\Pi^*), (T(\Pi^*))^*, T\left((T(\Pi^*))^*\right) \right). \quad (43)$$

**Definition 7.**

1. From Definitions 1 and 3 it follows that if  $\Pi$  is symmetric, then the numbers of the octet belong to the same interval  $\Omega_\nu$ .  $\Pi$  belongs to the octet, so  $\nu + 1 = \left\lceil \frac{\ln \Pi}{\ln 2} \right\rceil$ . We define this octet as *symmetric*.

2. From Definitions 1 and Theorem 8 it follows that if  $\Pi$  is asymmetric, then the numbers of the octet belong to the same interval  $\Omega_\nu$ . We define this octet as *asymmetric*.

We now give an example which also shows the ways in which we can write a symmetric octet.

**Example 8.** From Equation (43) we get the symmetric octet in which  $\Pi = 889$  belongs,  $\Phi = (889, 529, 1007, 895, 535, 1001, 641)$ . To distinguish the pairs of transposes and conjugates, we write the octet in the following form.

$$\begin{array}{ccccccc}
889 & \xleftarrow{T} & 529 & \xleftarrow{*} & 1007 & \xleftarrow{T} & 895 \\
*\updownarrow & & & & & & \updownarrow* \\
647 & \xleftarrow{T} & 535 & \xleftarrow{*} & 1001 & \xleftarrow{T} & 641
\end{array}$$

Because of Equation (13)  $(\Pi^*)^* = \Pi$  two conjugates are always connected, in all octets, by the symbol  $\xleftarrow{*}$ ,  $\Pi \xleftarrow{*} \Pi^*$ . With  $\Pi_1 \xleftarrow{T} \Pi_2$  we denote that  $T(\Pi_1) = \Pi_2$  and  $T(\Pi_2) = \Pi_1$ . If  $T(\Pi_1) = \Pi_2$  and  $T(\Pi_2) \neq \Pi_1$ , we write  $\Pi_1 \xrightarrow{T} \Pi_2$ . We follow this notation when  $\Pi_1$  is asymmetric. In our example the octet is symmetric. Therefore  $\Pi_1 \xleftarrow{T} \Pi_2$  is valid for all of the octet numbers. The octet symmetries are easily seen when we place the numbers on the corners of a regular octagon.

$$\begin{array}{ccc}
& 889 & \xleftarrow{T} & 529 & \\
& * \nearrow & & & \nwarrow * \\
647 & & & & 1007 \\
& & & & \\
T \updownarrow & & & & \updownarrow T \\
& & & & \\
535 & & & & 895 \\
& \nwarrow * & & & * \nearrow \\
& 1001 & \xleftarrow{T} & 641 &
\end{array}$$

A symmetric octet can be composed of eight different numbers, like the one of the previous example, or of 4 different numbers or of 2 different numbers (with the exception of the degenerate octets  $(1,1,1,1,1,1,1,1)$  of 1 and  $(3,3,3,3,3,3,3,3)$  of 3). From the Definitions of the conjugate and the transpose, the following equations are easily proven

$$\begin{aligned}
2^\nu + 1 &\xleftarrow{*} 2^{\nu+1} - 1 \\
2^\nu + 7 &\xleftarrow{*} 2^{\nu+1} - 7 \\
2^\nu + 1 &\xleftarrow{T} 2^{\nu+1} - 7 \\
2^\nu + 7 &\xleftarrow{T} 2^\nu + 7
\end{aligned} \tag{44}$$

$$2^{\nu+1} - 1 \xleftarrow{T} 2^{\nu+1} - 1$$

Considering Equations (44) we get the symmetric octets

$$(2^{\nu+1} + 1, 2^{\nu+2} - 7, 2^{\nu+1} + 7, 2^{\nu+1} + 7, 2^{\nu+2} - 1, 2^{\nu+2} - 1, 2^{\nu+1} + 1, 2^{\nu+2} - 7), \tag{45}$$

where  $\nu = 3, 4, 5, \dots$

The symmetric octets (45) consist of four different numbers. The Fermat numbers for  $\nu + 1 = 2^S$ ,  $S \in \mathbb{N}$ , and Mersenne numbers for  $\nu + 2 = p = \text{prime}$  belong to these octets. The symmetric octet  $(9, 9, 15, 15, 15, 15, 9, 9)$  of conjugates  $(\Pi, \Pi^*) = (9, 15)$  consists of two numbers.

**Asymmetric octets as generators of symmetric octets.** If an odd number  $\Pi$  belongs to a symmetric octet, then its conjugate  $\Pi^*$  and its transpose  $T(\Pi)$  belong to the octet. Also, all the numbers in the symmetric octet belong to

the same interval  $\Omega_\nu$ . The asymmetric octets result from a pair of conjugates  $(\Pi, \Pi^*)$  belonging to an interval  $\Omega_\nu$  and their transposes  $(T(\Pi), T(\Pi^*))$  in another interval  $\Omega_\mu$ ,  $\mu < \nu$  (refer to Theorem 8). The octet of the pair  $(T(\Pi), T(\Pi^*))$  is symmetric and we say that it is produced from the initial asymmetric octet.

We now present one example of an asymmetric octet in which one can see the way in which we can write it so that the asymmetry is evident and so are the symmetric octet that it produces.

**Example 9.** Let the pair of asymmetric conjugates  $(U = 10301, 14275 = V)$ . We have  $T(10301) = 641$  and  $T(14275) = 895$ . Thus  $U = 10301$  produces the symmetric octet to which  $Q = 641$  belongs, and  $V = 14275$  produces the (same) symmetric octet to which  $D = 895$  belongs.

$$\begin{array}{ccccccc}
10301 & & & & & & \\
\downarrow^T & & & & & & \\
641 & \xleftarrow{T} & 1001 & \xleftarrow{*} & 535 & \xleftarrow{T} & 647 \\
*\uparrow & & & & & & \downarrow^* \\
895 & \xleftarrow{T} & 1007 & \xleftarrow{*} & 529 & \xleftarrow{T} & 889 \\
\uparrow^T & & & & & & \\
14275 & & & & & & 
\end{array}$$

The conjugates numbers of the interval  $\Omega_\nu$  express a simple symmetry, they have a center of symmetry at the middle  $3 \cdot 2^\nu$  of the interval  $\Omega_\nu$ . As a symmetry, the transpose of a symmetric odd number  $\Pi$  ( $\Pi = Q$  or  $\Pi = D$ ) can be expressed through the Octet of Odd Numbers. Starting from a symmetric number  $\Pi$ , taking conjugate-transpose or transpose-conjugate we return to the number  $\Pi$ . A geometric interpretation of "transpose" is given by Proposition 7. "The conjugate" and "the transpose" are the fundamental symmetries that emerge from Theorem 1. By using these symmetries we can find a set of symmetries of the natural numbers.

## 6. Quadruples and pairs of odd numbers.

From the Definition 7 of symmetric octets and Theorem 3, we have that every symmetric octet consists of two ordered symmetric quadruples  $\Theta$  of the form

$$\Theta = (Q, Q + 6, Q^* - 6, Q^*), \quad (46)$$

so

$$\Phi = (Q_1, Q_1 + 6, Q_1^* - 6, Q_1^*, Q_2, Q_2 + 6, Q_2^* - 6, Q_2^*). \quad (47)$$

Quadruples (47) are symmetric, in the sense that they belong to symmetric octets. The differences of the corresponding numbers of two quadruples are the same. Thus we define the distance  $|h|$  of the quadruples of Equation (47) as

$$|h| = |Q_2 - Q_1|. \quad (48)$$

This equation also applies to all quadruplets, whether they belong to the same octet or not. If  $h \neq 0$ , then the symmetric octet consists of eight different numbers. If  $h = 0$ , then it consists of four different numbers. Such a quadruple is (45). In addition, if we use the equation  $Q + Q^* = 3 \cdot 2^{\nu+1}$ ,  $\nu \in \mathbb{N}$ , we take the asymmetric quadruple  $(Q = 3 \cdot 2^{\nu+1} - 3, Q + 6 = 3 \cdot 2^{\nu+1} + 3, Q^* - 6 = 3 \cdot 2^{\nu+1} - 3, Q^* = 3 \cdot 2^{\nu+1} + 3)$

which consists of the pair

$$(3 \cdot 2^{\nu+1} - 3, 3 \cdot 2^{\nu+1} + 3). \quad (49)$$

We prove the following.

**Proposition 8.**

1. If  $\nu = 1$ , the pair of  $\Omega_{\nu+1} = \Omega_2$  is

$$(Q = 9, D = 15).$$

2. If  $\nu \geq 2$ , the pair of the interval  $\Omega_{\nu+1}$  is of the form  $(U, V)$ ,

$$(U = 3 \cdot 2^{\nu+1} - 3, 3 \cdot 2^{\nu+1} + 3 = V).$$

3. The pair of asymmetric  $(U, V)$  produces the symmetric pair  $(9, 15)$  for every  $\nu \geq 2$ .

*Proof.* 1. If  $\nu = 1$ , from Equation (49) we get

$$(3 \cdot 2^2 - 3, 3 \cdot 2^2 + 3) = (9, 15).$$

2. We have,

$$(3 \cdot 2^{\nu+1} - 3) - 5 = 3 \cdot 2^{\nu+1} - 8 = 8 \cdot (2^{\nu-2} - 1) = 8m,$$

$$(3 \cdot 2^{\nu+1} + 3) - 3 = 3 \cdot 2^{\nu+1} = 8 \cdot 3 \cdot 2^{\nu-2} = 8m.$$

3. We have,

$$U = 3 \cdot 2^{\nu+1} - 3 = 2^{\nu+2} + 2^{\nu+1} - 2^{\nu} + 2^{\nu-1} + 2^{\nu-2} + 2^{\nu-3} + \dots + 2^1 - 1,$$

and from Equation (24) we get  $T(U) = 9$ . Similarly we get  $T(V) = 15$ .

□

If  $Q = 8m + 1$ ,  $m \in \mathbb{N}$ , belongs to the interval  $\Omega_{\nu} = [2^{\nu+1}, 2^{\nu+2}]$ ,  $\nu = 3, 4, 5, \dots$ , then  $Q, Q + 6$  belong to the interval  $[2^{\nu+1}, 3 \cdot 2^{\nu}]$  and their conjugates  $Q^*, Q^* - 6$  belong to the interval  $[3 \cdot 2^{\nu}, 2^{\nu+2}]$ . Therefore, the different octets of the interval  $\Omega_{\nu}$  are given by the inequality

$$2^{\nu+1} + 1 \leq 8m + 1 < 3 \cdot 2^{\nu} + 1$$

or equivalently

$$2^{\nu-2} \leq m < 3 \cdot 2^{\nu-3}.$$

From this inequality we get

$$Q = 8(2^{\nu-2} + k) + 1, \quad k = 0, 1, 2, \dots, 2^{\nu-3} - 1. \quad (50)$$

where  $Q = Q_1$  or  $Q = Q_2$ ,  $Q_1, Q_2 \in \Omega_{\nu}$ .

From Equation (50) we get the following.

**Corollary 13.**

*Interval  $\Omega_{\nu}$  contains exactly  $2^{\nu-3}$  different symmetric quadruples.*

Now let the odd numbers  $Q$  of the quadruple (46) with (left) symmetry  $L \geq 4$ ,

$$Q = 2^L \cdot K + 1, \quad L \geq 3, \quad (51)$$

where  $K$  is an odd number. Taking into account that the quadruples (46) belong to the interval

$[2^{\nu+1} + 1, 3 \cdot 2^{\nu} + 1)$  we have  $2^{\nu+1} + 1 \leq 2^L \cdot K + 1 < 3 \cdot 2^{\nu} + 1$  or equivalently  $2^{\nu+1-L} < K < 3 \cdot 2^{\nu-L}$  and finally we obtain

$$K = 2^{\nu+1-L} + 1, 2^{\nu+1-L} + 3, 2^{\nu+1-L} + 5, \dots, 3 \cdot 2^{\nu-L} - 1.$$

From this Equation we get

$$K = 2^{\nu+1-L} + 1 + 2\lambda, \quad \lambda = 0, 1, 2, \dots, 2^{\nu-1-L} - 1, \quad \nu \geq L + 1. \quad (52)$$

From Equations (51) and (52) we obtain,

$$Q_{\lambda} = 2^L \cdot (2^{\nu+1-L} + 1 + 2\lambda) + 1, \quad L \geq 3, \quad \nu \geq L + 1, \quad \lambda = 0, 1, 2, \dots, 2^{\nu-L} - 1. \quad (53)$$

From Equation (53) it follows that if  $\nu \geq L + 1$ , the interval  $\Omega_{\nu}$  contains exactly  $N = 2^{\nu-L}$  different symmetric quadruples with symmetry  $L$ .

There are quadruples of odd numbers containing asymmetric numbers.

**Definition 8.**

We define as *asymmetric* the quadruples

$$(U, U+4, U^*-4, U^*), (V, V+4, V^*-4, V^*).$$

Numbers  $U+4$  and  $V^*-4$  are of the form  $Q$ . Numbers  $U^*-4$  and  $V+4$  are of form  $D$ . For the asymmetric numbers  $U$  and  $V$  we have,

$$U \xrightarrow{T} Q$$

and

$$V \xrightarrow{T} D.$$

We now prove the following.

**Proposition 9.**

1. If one of the equations

$$U = 2^n \cdot (Q+3) - 3 \xrightarrow{T} T(Q) \xleftarrow{T} Q,$$

$$V = 2^n \cdot (D-3) + 3 \xrightarrow{T} T(D) \xleftarrow{T} D$$

is valid, then the other is also valid,  $n = 0, 1, 2, \dots$

2. If one of the equations

$$U' = 2^n \cdot (D-3) - 3 \xrightarrow{T} T((T(D))^*) \xleftarrow{T} (T(D))^*,$$

$$V' = 2^n \cdot (Q+3) + 3 \xrightarrow{T} T((T(Q))^*) \xleftarrow{T} (T(Q))^*$$

is valid, then the other is also valid,  $n = 0, 1, 2, \dots$

*Proof.* We prove one of the combinations. The remaining combinations prove similarly. We assume that the third Equation is valid, we put  $D = Q^*$  and we get

$$V = 2^n \cdot (Q^* - 3) + 3 \xrightarrow{T} T(Q^*) \xleftarrow{T} Q^*,$$

or equivalently, changing the symbolism,

$$T(2^n \cdot (Q^* - 3) + 3) = T(Q^*). \tag{54}$$

From Equation (6) it follows that if  $Q \in \Omega_\nu$ , then  $U \in \Omega_{\nu+n}$ . Thus, from Proposition 7 we get

$$T(U) = T(3 \cdot 2^{n+\nu+1} - U) - 6$$

or equivalently

$$T(U) = T(3 \cdot 2^{n+\nu+1} - 2^n \cdot (Q+3) + 3) - 6$$

or equivalently

$$T(U) = T(2^n \cdot (3 \cdot 2^{\nu+1} - Q - 3) + 3) - 6$$

or equivalently

$$T(U) = T(2^n \cdot (Q^* - 3) + 3) - 6.$$

and with Equation (54) we get

$$T(U) = T(Q^*) - 6$$

or equivalently, changing the symbolism,

$$U \xrightarrow{T} T(Q^*) - 6$$

and considering Equation (46) for the quadruples of odd numbers ( $D-6 = Q$ , where  $D = T(Q^*)$ ) we obtain,

$$U \xrightarrow{T} T(Q^*) - 6 \xleftarrow{T} T(Q) \xleftarrow{T} Q.$$

□

From Equations (46), (47) and (14) we get the following.

**Proposition 10.**

1. The numbers  $Q_1$  and  $Q_2 = 3 \cdot 2^{v+1} - 6 - Q_1$  of the interval  $\Omega_v$  belong to the same symmetric octet.
2. The numbers  $D_1$  and  $D_2 = 3 \cdot 2^{v+1} + 6 - D_1$  of the interval  $\Omega_v$  belong to the same symmetric octet.

From Proposition 10 we get the following.

**Corollary 14.**

- A. 1. Half of the numbers in a symmetric octet have  $L/R$  symmetry with value  $n = 3$ , and the other half have symmetry  $n > 3$ .
2. In a symmetric octet,  $K$  takes four different values. These values are calculated from the equations  $Q - 1 = 2^n \cdot K$  for the odd numbers of form  $Q$  of the octet, and  $D - 7 = 2^n \cdot K$  for the odd numbers of form  $D$ .
- B. 1. Half of the numbers in a quadruple have  $L/R$  symmetry with value  $n = 3$ , and the other half have symmetry  $n > 3$ .
2. In a quadruple,  $K$  takes two different values. These values are calculated from the equations  $Q - 1 = 2^n \cdot K$  for the odd numbers of form  $Q$  of the octet, and  $D - 7 = 2^n \cdot K$  for the odd numbers of form  $D$ .

**7. Structural equations of the odd number transpose**

In the list below we have the transposes of eight odd numbers. The list shows the structure of the odd numbers related through the symmetry of the 'transpose'.

**List of characteristic transposes.**

$$N \in \mathbb{N}, N \geq 3$$

$$U = 2^N - 3 \xrightarrow{T} 1$$

$$Q = 2^N + 1 \xleftarrow{T} 2^{N+1} - 7$$

$$V = 2^N + 3 \xrightarrow{T} 7$$

$$U = 2^N + 5 \xrightarrow{T} 2^N - 7$$

$$V = 2^N - 5 \xrightarrow{T} 2^{N-1} - 1$$

$$D = 2^N + 7 \xleftarrow{T} 2^N + 7$$

$$Q = 2^N - 7 \xleftarrow{T} 2^{N-1} + 1$$

$$V = 3 \cdot (2^N + 1) \xrightarrow{T} 15$$

$$U = 5 \cdot (2^N + 1) \xrightarrow{T} 2^{N+2} - 23$$

$$D = 7 \cdot (2^N + 1) \xleftarrow{T} 2^{N+2} + 31$$

In the list the consequences of Theorem 8 for the asymmetric  $U$  and  $V$  are seen. Asymmetric  $U = 2^N + 5$  has the smallest possible difference of an asymmetric with its transpose,

$$2^N + 5 - T(2^N + 5) = 2^N + 5 - (2^N - 7) = 5 - (-7) = 12.$$

However, as a consequence of Theorem 8,

$$U \in \Omega_{N-1} = [2^N, 2^{N+1}] \text{ and } T(U) \in \Omega_{N-2} = [2^{N-1}, 2^N].$$

We now give the two techniques for calculating the transposes of odd numbers of the form  $2^N - \Pi \geq 1$  and  $2^N + \Pi$ , where  $\Pi$  is an odd number and  $N \in \mathbb{N}$ . With these techniques, Equations of the list of characteristic transposes are calculated. From the list of characteristic transposes, we prove the following.

$$2^N - 7 \xrightarrow{T} 2^{N-1} + 1$$

For  $2^N - 7$  we have the following sequence of equations

$$2^N - 7 = 4 \cdot 2^{N-2} - 7$$

$$2^N - 7 = 3 \cdot 2^{N-2} + 2^{N-2} - 7$$

$$2^N - 7 = 2^{N-1} + 2^{N-2} + 2^{N-2} - 1 - 2 - 2^2$$

$$2^N - 7 = 2^{N-1} + 2^{N-2} + (2^{N-2} - 1) - 2 - 2^2$$

$$2^N - 7 = 2^{N-1} + 2^{N-2} + (2^{N-3} + 2^{N-4} + 2^{N-5} + \dots + 1) - 2 - 2^2$$

$$2^N - 7 = 2^{N-1} + 2^{N-2} + 2^{N-3} + 2^{N-4} + 2^{N-5} + \dots + 2^2 + 2 + 1 - 2 - 2^2$$

$$2^N - 7 = 2^{N-1} + 2^{N-2} + 2^{N-3} + 2^{N-4} + 2^{N-5} + \dots + 2^2 - 2 - 1$$

and from Equation (22) we obtain

$$T(2^N - 7) = -(1 + 2 + 2^2 + \dots + 2^{N-3} - 2^{N-2} - 2^{N-1}) = 2^{N-1} + 1.$$

We now prove the following.

$$2^N + 7 \xrightarrow{T} 2^N + 7$$

For  $2^N + 7$  we have the following sequence of equations

$$2^N + 7 = 2^N + 1 + 2^2 + 2$$

$$2^N + 7 = 2^N + (2^{N-1} - 2^{N-2} - 2^{N-3} - 2^{N-4} - \dots - 2^2 - 2 - 1) + 2^2 + 2$$

$$2^N + 7 = 2^N + 2^{N-1} - 2^{N-2} - 2^{N-3} - 2^{N-4} - \dots - 2^2 - 2 - 1 + 2^2 + 2$$

$$2^N + 7 = 2^N + 2^{N-1} - 2^{N-2} - 2^{N-3} - 2^{N-4} - \dots - 2^2 + 2 + 1$$

and from Equation (20) we obtain

$$T(2^N + 7) = 2^N + 7.$$

Combining the list of characteristic transposes with Theorem 9 we obtain a set of structural equations for the odd number transpose. We give four such combinations.

For

$$A = 2^N - 7 = 2^{N-1} + 2^{N-2} + 2^{N-3} + 2^{N-4} + 2^{N-5} + \dots + 2^2 - 2 - 1, \quad N \geq 4,$$

we have  $\nu = N - 2$  and  $\beta_2 = \beta_3 = \beta_4 = \dots = \beta_{N-3} = +1$ . Thus from Equation (42) we get

$$T(2^N - 7 - 2^{k+1}) - T(2^N - 7) = 2^{(N-2)+2-k}$$

and with the equation  $T(2^N - 7) = 2^{N-1} + 1$  we get

$$T(2^N - 7 - 2^{k+1}) = 2^{N-k} + 2^{N-1} + 1, \quad k = 2, 3, 4, \dots, N-3. \quad (55)$$

We have  $\beta_1 = -1$ . Thus from Equations (42) and  $T(2^N - 7) = 2^{N-1} + 1$  we get

$$T(2^N - 7 + 2^2) - (2^{N-1} + 1) = -2^{(N-2)+2-1}$$

or equivalently  $T(2^N - 3) = 1$ .

Working similarly, from

$$\Delta = 2^N + 7 = 2^N + 2^{N-1} - 2^{N-2} - 2^{N-3} - 2^{N-4} - \dots - 2^2 + 2 + 1, \quad N \geq 4,$$

$$\Delta = 2^{N+2} - 1 = 2^{N+1} + 2^N + 2^{N-1} + \dots + 2^2 + 2 + 1, \quad N \geq 4,$$

$$A = 2^N - 3 = 2^{N-1} + 2^{N-2} + 2^{N-3} + \dots + 2^2 + 2 - 1, \quad N \geq 4,$$

we obtain the following equations.

$$T(2^N + 7 + 2^{k+1}) = 2^N + 7 + 2^{N+1-k}, \quad k = 2, 3, 4, \dots, N-2, \quad (56)$$

$$T(2^N - 1 - 2^{k+1}) = 2^N - 1 - 2^{N-k}, \quad k = 1, 2, 3, \dots, N-3, \quad (57)$$

$$T(2^N - 3 - 2^{k+1}) = 2^{N-k} + 1, \quad k = 1, 2, 3, \dots, N-3. \quad (58)$$

## 8. The sequences $A(N)$ and $B(N)$

Proposition 5 and Corollaries 10, 11, 12 give the dependence of the symmetry  $L/R$  of the product  $\Pi_1 \Pi_2$  on the symmetry of  $\Pi_1$  and  $\Pi_2$ . Of all the possible forms that the factors of the sequences

$$A(N) = 2^N + 1$$

and

$$B(N) = 2^N - 1$$

can have, one form they always, necessarily, take is the form of the factors of Corollary 2. This has consequences for the factors of the sequences  $A(N)$  and  $B(N)$ , some of which we will see next.

### Theorem 10.

*The sequences  $A(N)$  and  $B(N)$  have no common factor.*

*Proof.* The theorem is a consequence of equation  $A(N) + B(N) = 2^{N+1}$ .

□

We present two Theorems for the sequences  $A(N)$  and  $B(N)$ . The extremely large amount of information contained in Proposition 5 and Corollaries 9, 10, 11 makes their proof difficult. Thus, out of the total of 12 Theorems we have presented so far on the symmetries of odd numbers, two of them, Theorems 11 and 12, have not been fully proven.

### Theorem 11.

1. If  $N$  is an odd number, then the prime factors of sequence  $A(N) = 2^N + 1$  are either of form  $Q$  or of form  $V$ .
2. If  $N \neq 1$  is an odd number, then the sequence  $B(N) = 2^N - 1$  has no prime asymmetric factors.

### Theorem 12.

1. If  $P \geq 5$  is a prime factor of  $A(N) = 2^N + 1$ ,  $N \in \mathbb{N}^*$ , then  $P-1$  and  $N$  have at least one common factor.
2. If  $P$  is a prime factor of  $B(N) = 2^N - 1$ ,  $N \in \mathbb{N}^*$ , then  $P-1$  and  $N$  have at least one common factor.

From Theorem 11 and Proposition 5 we get the following.

**Corollary 15.**

1. If  $N \neq 1$  is an odd number, then the sequence  $A(N) = 2^N + 1$  has an even number of prime factors of the form  $V$ .
2. If  $N \neq 1$  is an odd number, then the sequence  $B(N) = 2^N - 1$  has no asymmetric factors.
3. If  $N$  is an odd number, then the sequences  $A(N) = 2^N + 1$  and  $B(N) = 2^N - 1$  have no factors of the form  $U$ .

We now prove the following.

**Proposition 11.**

If  $q$  is a prime odd number, then the following holds.

1. All prime factors  $P > 5$  of  $A(q) = 2^q + 1$  are of the form  $P = 2^m \cdot kq + 1$  (equivalent to  $P - 1 = 2^m \cdot kq$ ), where  $k$  is an odd number and  $m \in \mathbb{N}^*$ .
2. All prime factors  $P$  of  $B(q) = 2^q - 1$  are of the form  $P = 2^m \cdot kq + 1$  (equivalent to  $P - 1 = 2^m \cdot kq$ ), where  $k$  is an odd number and  $m \in \mathbb{N}^*$ .

*Proof.* We prove 1. The proof of 2 is similar. From Theorem 12 we get that if  $P > 5$  is a prime factor of  $A(q) = 2^q + 1$  then  $P - 1$  has at least one common factor with  $q$ , and since  $q$  is prime number  $P - 1$  has a factor the  $q$ . Taking into account that  $P - 1$  is an even number we get  $P - 1 = 2^m \cdot kq$ , where  $k$  is an odd number and  $m \in \mathbb{N}^*$ . □

From Proposition 11 we get the following.

**Corollary 16.**

1. If  $q$  is a prime number, then

$$A(q) = 2^q + 1 = 3 \cdot \prod_{i=1}^N (2^{N_i} k_i q + 1),$$

where  $k_i$  is an odd number and  $N, N_i \in \mathbb{N}^*$ .

2. If  $q$  is a prime number, then

$$B(q) = 2^q - 1 = \prod_{i=1}^Z (2^{Z_i} l_i q + 1),$$

where  $l_i$  is an odd number and  $Z, Z_i \in \mathbb{N}^*$ .

We prove the following, for the form of the sequences  $A(N)$  and  $B(N)$  given by Corollary 2.

**Proposition 12.**

1. If  $\Pi \neq 1$  is an composite odd number, then in Equation

$$A(\Pi) = 2^\Pi + 1 = (2^x \cdot K_1 + 1)(2^x \cdot K_2 + 1)$$

, the odd numbers  $K_1$  and  $K_2$  cannot have a common factor.

2. If  $\Pi \neq 1$  is an composite odd number, then in Equation

$$B(\Pi) = 2^\Pi - 1 = (2^y \cdot K_3 + 1)(2^y \cdot K_4 - 1)$$

, the odd numbers  $K_3$  and  $K_4$  cannot have a common factor.

*Proof.* We prove 1. 2 is proved similarly. If  $K_1 = k_1 a$  and  $K_2 = k_2 a$ , where  $a \neq 1$ ,  $k_1, k_2$  are odd numbers, we have

$$A(\Pi) = 2^\Pi + 1 = (2^x \cdot k_1 a + 1)(2^x \cdot k_2 a + 1)$$

or equivalent

$$2^\Pi = 2^{2^x} \cdot k_1 k_2 a^2 + 2^x \cdot k_1 a + 2^x \cdot k_2 a$$

or equivalent

$$2^\Pi = 2^x \cdot a(2^x \cdot k_1 k_2 a + k_1 + k_2).$$

This equation is impossible, since  $2^\Pi$  cannot have the odd number  $a \neq 1$  as a factor. □

From Theorem 12 and Proposition 12 we get the following.

**Corollary 17.**

*Let  $\Pi$  be a composite odd number and*

$$A(\Pi) = 2^\Pi + 1,$$

$$B(\Pi) = 2^\Pi - 1.$$

*$A(\Pi)$  and  $B(\Pi)$  have prime factors  $P > 5$  for which  $P-1$  has at least one common factor with  $\Pi$ , while it does not have  $\Pi$  itself as a factor.*

From Corollary 17 we conclude that there are prime factors  $P > 5$  of  $A(\Pi) = 2^\Pi + 1$  and  $B(\Pi) = 2^\Pi - 1$  which give the factors of  $\Pi$ .

We now give two examples.

**Example 10.**

For  $N = 77 = 7 \cdot 11$  we have

$$A(N) = 2^{77} + 1 = 3 \cdot 43 \cdot 617 \cdot 683 \cdot 78233 \cdot 35532364099.$$

$N = 7 \cdot 11$  is an odd number. As Corollary 15 predicts,  $A(77)$  has an even number of prime asymmetric factors of the form  $V$ , namely 3, 43, 683, 35532364099. As Theorem 11 predicts, the other two factors 617 and 78233 of  $A(77)$  are of form  $Q$ .

Additional, as provided for by Corollary 17, for the prime factors of  $A(7 \cdot 11)$  we have

$$43 - 1 = 2 \cdot 3 \cdot 7,$$

$$617 - 1 = 2^3 \cdot 7 \cdot 11,$$

$$683 - 1 = 2 \cdot 11 \cdot 31,$$

$$78233 - 1 = 2^3 \cdot 7 \cdot 11 \cdot 127,$$

$$35532364099 - 1 = 2 \cdot 3 \cdot 7 \cdot 11 \cdot 229 \cdot 263 \cdot 1277.$$

Factor 7 of  $\Pi = 7 \cdot 11$  appears in the first, second, fourth, and fifth equations. The factor 11 of  $\Pi = 7 \cdot 11$  appears in the second, third, fourth, and fifth equations. In the first equation, only 7 appears, and in the third only 11.  $N = 7 \cdot 11$  itself appears in the second, fourth, and fifth equations.

For  $B(N)$  we have

$$B(N) = 2^{77} - 1 = 23 \cdot 89 \cdot 127 \cdot 581283643249112959.$$

$N = 7 \cdot 11$  is an odd number. As Corollary 15 predicts,  $B(7 \cdot 11)$  has no asymmetric factors. Indeed,  $23 = D_1$ ,  $89 = Q$ ,  $127 = D_2$  are symmetric numbers, as are all their possible products, as a consequence of Proposition 5.

Additional, as provided for by Corollary 17, for the prime factors of  $B(7 \cdot 11)$  we have

$$23 - 1 = 2 \cdot 11,$$

$$89 - 1 = 2^3 \cdot 11,$$

$$127 - 1 = 2 \cdot 3^2 \cdot 7,$$

$$581283643249112959 - 1 = 2 \cdot 3^2 \cdot 7 \cdot 11 \cdot 31 \cdot 13528921548413.$$

Factor 11 of  $N = 7 \cdot 11$  appears in the first two equations. The factor 7 of  $N = 7 \cdot 11$  appears in the third equation. In the fourth equation  $N = 7 \cdot 11$  itself appears.

As Corollary 15 predicts, none of the factors of sequences  $A(77)$  and  $B(77)$ , prime or composite, are of the form  $U$ .

**Example 11.**

For the prime number 97 we have

$$A(97) = 2^{97} + 1 = 3 \cdot (2^1 \cdot 5 \cdot 97 + 1) (2^4 \cdot 97 + 1) (2^3 \cdot 41 \cdot 97 + 1) (2^2 \cdot 1418\ 654662\ 841345 \cdot 97 + 1)$$

and

$$B(97) = 2^{97} - 1 = (2^1 \cdot 59 \cdot 97 + 1) (2^3 \cdot 17838411386376914491967 \cdot 97 + 1), \text{ as predicted by Corollary 16.}$$

**9. The  $K$  numbers.**

We now give the following definition.

**Definition 9.**

We define as “the  $K$  numbers” of an odd number  $\Pi > 7$  the odd numbers  $K_1, K_2, K_3, K_4$  given by the equations

$$\Pi - 1 = 2^{N_1} \cdot K_1,$$

$$\Pi - 3 = 2^{N_2} \cdot K_2,$$

$$\Pi - 5 = 2^{N_3} \cdot K_3,$$

$$\Pi - 7 = 2^{N_4} \cdot K_4,$$

where  $N_1, N_2, N_3, N_4 \in \mathbb{N}^*$ .

From Definition 9 we get the following.

**Corollary 18.**

1. Each of the numbers  $Q \neq 1, V \neq 3, U \neq 5, D \neq 7$  is written in the following four forms.

$$Q = 2^n \cdot K_1 + 1 = 2 \cdot K_2 + 3 = 2^2 \cdot K_3 + 5 = 2 \cdot K_4 + 7$$

$$V = 2 \cdot K_1 + 1 = 2^n \cdot K_2 + 3 = 2 \cdot K_3 + 5 = 2^2 \cdot K_4 + 7$$

$$U = 2^2 \cdot K_1 + 1 = 2 \cdot K_2 + 3 = 2^n \cdot K_3 + 5 = 2 \cdot K_4 + 7$$

$$D = 2 \cdot K_1 + 1 = 2^2 \cdot K_2 + 3 = 2 \cdot K_3 + 5 = 2^n \cdot K_4 + 7$$

$$n \in \mathbb{N}, n \geq 3$$

2. For the same value of  $n$ ,  $Q, V, U, D$  are consecutive odd numbers.

We prove the following.

**Theorem 13.**

1. It is impossible for two different odd numbers of the form  $Q$  to have  $K_2$  or  $K_3$  or  $K_4$  equal.

2. It is impossible for two different odd numbers of the form  $V$  to have  $K_1$  or  $K_3$  or  $K_4$  equal.

3. It is impossible for two different odd numbers of the form  $U$  to have  $K_1$  or  $K_2$  or  $K_4$  equal.

4. It is impossible for two different odd numbers of the form  $D$  to have  $K_1$  or  $K_2$  or  $K_3$  equal.

*Proof.* We prove the Theorem for odd numbers of form  $Q$ . The proof for odd numbers of form  $V, U$ , and  $D$  is similar. If  $Q_1 \neq Q_2$ , then Corollary 19 implies that  $Q_1$  and  $Q_2$  cannot have any of  $K_2, K_3, K_4$  in common. Indeed, suppose that  $Q_1$  and  $Q_2$  have the same  $K_2$ . In this case we have  $Q_1 = 2K_2 + 1 = Q_2$  which is impossible since  $Q_1 \neq Q_2$ . We arrive at the same conclusion if we assume that  $Q_1$  and  $Q_2$  have equal  $K_3$  or  $K_4$ .

□

From Theorem 13 the following follows.

**Corollary 19.**

1. Two different odd numbers of form  $Q$ , from  $K_1, K_2, K_3, K_4$ , can have only the  $K_1$  in common (having different  $N_1 \geq 3$ ).
2. Two different odd numbers of form  $V$ , from  $K_1, K_2, K_3, K_4$ , can have only the  $K_2$  in common (having different  $N_2 \geq 3$ ).
3. Two different odd numbers of form  $U$ , from  $K_1, K_2, K_3, K_4$ , can have only the  $K_3$  in common (having different  $N_3 \geq 3$ ).
4. Two different odd numbers of form  $D$ , from  $K_1, K_2, K_3, K_4$ , can have only the  $K_4$  in common (having different  $N_4 \geq 3$ ).

**Corollary 20.**

1. The  $K$  numbers of  $A(N) = 2^N + 1$  are as follows.

$$K_1 = 1$$

$$K_2 = 2^{N-1} - 1$$

$$K_3 = 2^{N-2} - 1$$

$$K_4 = 2^{N-1} - 3$$

2. The  $K$  numbers of  $B(N) = 2^N - 1$  are as follows.

$$K_1 = 2^{N-1} - 1$$

$$K_2 = 2^{N-2} - 1$$

$$K_3 = 2^{N-1} - 3$$

$$K_4 = 2^{N-3} - 1$$

For the numbers  $K'$  of the factors  $a_j$  of  $K_j$ ,  $j = 1, 2, 3, 4$ , we follow the following symbolism.

$$a_j - 1 = 2^{M_1} \cdot K'_{j1}(a_j) = 2^{M_1} \cdot K'_{j1}$$

$$a_j - 3 = 2^{M_2} \cdot K'_{j2}(a_j) = 2^{M_2} \cdot K'_{j2}$$

$$a_j - 5 = 2^{M_3} \cdot K'_{j3}(a_j) = 2^{M_3} \cdot K'_{j3}$$

$$a_j - 7 = 2^{M_4} \cdot K'_{j4}(a_j) = 2^{M_4} \cdot K'_{j4}$$

$$M_1, M_2, M_3, M_4 \in \mathbb{N}^*, j = 1, 2, 3, 4$$

The odd number  $K$  in Proposition 3 plays an important role in the structure of an odd number. This is clearly seen in the structure of the factors of Fermat numbers [1 - 5]. Below we will see that in the structure of an odd number all of  $K_1, K_2, K_3$  and  $K_4$  play a role, and not just  $K$  of Proposition 3, which is one of them.

## 10. An algorithm for factoring odd numbers

In this Section we apply the theory we presented to the factorization of odd numbers. The well-known factorization tests aim at finding the factors of an integer [6 - 11]. The algorithm for factoring an odd number  $\Pi$  that arises from the symmetries of the natural numbers aims at finding an octet of odd numbers to which a factor of  $\Pi$  belongs. We present the steps of algorithm.

*Step 1.* Calculate  $K_1, K_2, K_3, K_4$  of the odd number  $\Pi$ .

Step 2. Calculate the numbers  $K'$  of factors of  $K_1, K_2, K_3$  and  $K_4$  (or of  $K_1, K_2, K_3, K_4$  themselves). At least one factor of  $K'$  belongs to the same octet as a factor of  $\Pi$ . Thus, from factors of  $K_1, K_2, K_3, K_4$  we find factors of  $\Pi$ .

We give two examples.

**Example 12.**

For the odd number  $\Pi = 46793$  we have

$$\Pi - 1 = 2^3 \cdot 5849$$

$$\Pi - 3 = 2 \cdot 5 \cdot 4679$$

$$\Pi - 5 = 2^2 \cdot 3 \cdot 7 \cdot 557$$

$$\Pi - 7 = 2 \cdot 149 \cdot 157$$

therefore we have

$$K_1 = 5849$$

$$K_2 = 5 \cdot 4679$$

$$K_3 = 3 \cdot 7 \cdot 557$$

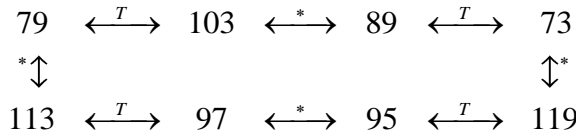
$$K_4 = 149 \cdot 157$$

From the factor 5849 of  $K_1$  we get  $5849 - 3 = 2 \cdot 37 \cdot 79 = 2 \cdot K'_{12}$ .

From the factor 4679 of  $K_2$  we get  $4679 - 7 = 2^6 \cdot 73 = 2^6 \cdot K'_{24}$ .

From the factor 149 of  $K_4$  we get  $149 - 3 = 2 \cdot 73 = 2 \cdot K'_{32}$ .

Factors 79 of  $K'_{12} = 37 \cdot 79$ , 73 of  $K'_{24} = 73$  and 73 of  $K'_{32} = 73$  give the octet to which the factor 73 of  $\Pi = 46793 = 73 \cdot 641$  belongs.



Every odd number, symmetric or asymmetric, belongs to its octet. This explains the appearance of the factor 73 of  $\Pi = 73 \cdot 641$  in  $K'_{24}$  and  $K'_{32}$ .

**Example 13.**

For the fifth Fermat number  $F_5 = 2^{32} + 1$  we have

$$F_5 - 1 = 2^{32} \cdot 1$$

$$F_5 - 3 = 2 \cdot 2147483647$$

$$F_5 - 5 = 2^2 \cdot 3^2 \cdot 7 \cdot 11 \cdot 31 \cdot 151 \cdot 331$$

$$F_5 - 7 = 2 \cdot 5 \cdot 19 \cdot 22605091$$

therefore we have

$$K_1 = 1$$

$$K_2 = 2147483647$$

$$K_3 = 3^2 \cdot 7 \cdot 11 \cdot 31 \cdot 151 \cdot 331$$

$$K_4 = 5 \cdot 19 \cdot 22605091$$

From the factor  $3 \cdot 7 \cdot 31 \cdot 151 \cdot 331$  of  $K_3$  we get

$$3 \cdot 7 \cdot 31 \cdot 151 \cdot 331 - 1 = 2 \cdot 47 \cdot 535 \cdot 647 = 2 \cdot K'_{31}$$

The factors 535 and 647 of  $K'_{31} = 47 \cdot 535 \cdot 647$  give the octet to which the factor 641 of  $F_5 = 641 \cdot 6700417$  belongs (refer to Example 8).

In the case where  $\Pi$  has an asymmetric factor,  $V$  or  $U$ , there is a possibility that the algorithm will find a symmetric number of the octet produced by the asymmetric factor of  $\Pi$  (and not the asymmetric factor  $V$  or  $U$  of  $\Pi$  itself). In this case the asymmetric factor of  $\Pi$  is found by Proposition 9. We give an example.

**Example 13.**

For the odd number  $\Pi = 1413787987 = 211 \cdot 6700417$  we have

$$\Pi - 1 = 2^1 \cdot 3^3 \cdot 13 \cdot 19 \cdot 105997$$

$$\Pi - 3 = 2^4 \cdot 7^2 \cdot 31 \cdot 58171$$

$$\Pi - 5 = 2^2 \cdot 37 \cdot 19105243$$

$$\Pi - 7 = 2^1 \cdot 3 \cdot 5 \cdot 11 \cdot 607 \cdot 3529$$

therefore we have

$$K_1 = 3^3 \cdot 13 \cdot 19 \cdot 105997$$

$$K_2 = 7^2 \cdot 31 \cdot 58171$$

$$K_3 = 37 \cdot 19105243$$

$$K_4 = 3 \cdot 5 \cdot 11 \cdot 607 \cdot 3529$$

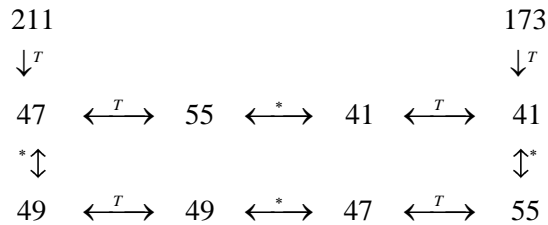
From the factor  $13 \cdot 19$  of  $K_1$  we get  $3 \cdot 19 - 1 = 2 \cdot 3 \cdot 41 = 2K'_{11}$ .

From the factor  $7 \cdot 58171$  of  $K_2$  we get  $7 \cdot 58171 - 7 = 3 \cdot 5 \cdot 49 \cdot 277 = 2 \cdot K'_{24}$ .

From the factor  $19105243$  of  $K_3$  we get  $19105243 - 3 = 2^3 \cdot 7 \cdot 55 \cdot 6203 = 2^3 \cdot K'_{32}$ .

From the factor  $3 \cdot 5 \cdot 11$  of  $K_4$  we get  $3 \cdot 5 \cdot 11 - 1 = 2^2 \cdot 41 = 2^2 \cdot K'_{41}$ .

Factor 41 of  $K'_{11} = 3 \cdot 41$ , factor 49 of  $K'_{24} = 3 \cdot 5 \cdot 49 \cdot 277$ , factor 55 of  $K'_{32} = 7 \cdot 55 \cdot 6203$ , and factor 41 of  $K'_{41} = 41$  belong to the octet produced by the asymmetric factor  $211 = V$  of  $\Pi = 211 \cdot 6700417$ .



The asymmetric factor  $V = 211$  of  $\Pi = 211 \cdot 6700417$  is found by the symmetric number  $D = 55$  of octet, by Proposition 9.

$$2^2 \cdot (55 + 3) - 3 = 211$$

For a given odd number  $\Pi$ , the efficiency of the algorithm depends on two factors. The first factor concerns the possibility of factoring the numbers  $K$  and  $K'$ . In order to run the algorithm, knowledge of these factors is required. The numbers  $K$  or  $K'$  that are factored by Algebraic Identities accelerate the application of the algorithm. The numbers  $K$  or  $K'$  that are not factored by Identities can be factored by applying the algorithm to  $K$  or  $K'$  themselves. However, in the application of the algorithm, not all factors of  $K$  and  $K'$  contribute. Thus, the algorithm can be applied even in cases where we do not have a complete factorization of them. The second factor concerns the number of factors of  $K$  and  $K'$ . This number determines the number of tests required to complete the algorithm. As the value of  $\Pi$  increases, the number of factors of  $K$  and  $K'$ , and therefore the number of tests required to complete the algorithm, can either increase or decrease.

For the Fermat numbers  $F_s$ ,

$$F_s = A(2^s) = 2^{2^s} + 1, S \in \mathbb{N},$$

we have  $N = 2^s$  in 1 of Corollary 20. Thus we get

$$K_1 = 1$$

$$K_2 = 2^{2^s-1} - 1$$

$$K_3 = 2^{2^s-2} - 1$$

$$K_4 = 2^{2^s-1} - 3$$

If the odd number  $2^s - 1$  in  $K_2$  is not prime, the factors of  $K_2$  are found by Algebraic Identities. The number  $2^s - 2$  in  $K_3$  is even, therefore the factors of  $K_3$  are found by Algebraic Identities.

For the Mersenne numbers

$$B(P) = 2^P - 1,$$

where  $P$  is a prime number, we have  $N = P$  in 2 of Corollary 20. Thus, we get

$$K_1 = 2^{P-1} - 1$$

$$K_2 = 2^{P-2} - 1$$

$$K_3 = 2^{P-1} - 3$$

$$K_4 = 2^{P-3} - 1$$

The number  $P-1$  in  $K_1$  is even, so the factors of  $K_1$  are found by Algebraic Identities. If the odd number  $P-2$  is not prime, the factors of  $K_2$  are found by Algebraic Identities. The number  $P-3$  in  $K_4$  is even, so the factors of  $K_4$  are found by Algebraic Identities.

For odd numbers  $\Pi$  of a specific form, such as the Fermat and Mersenne numbers, the numbers  $K$  are determined which can be factored by Algebraic Identities. These numbers speed up the factorization algorithm of  $\Pi$ .

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