The Geodesic Principle and the Nature of Passive Mass

R. Wolnik M. Wolnik

Email: wolnik@akademiarac.edu.pl

Abstract

The geodesic principle represents an essential aspect of general relativity and is the physical manifestation of the space-time manifold but can also be considered as the metric field effect on the passive mass of a freely falling test particle. – The equation of motion is derived from the given stress-energy tensor field of an isolated body, with the help of its moments in the near limit case, on the basis of the covariant conservation condition. Then, the reduced stress-energy tensor, grounded in the spatial energy density of the body, is used in the context of its total energy balance to obtain the global solution in the form of the geodesic equation. Finally, the potential implications for understanding the background of the weak equivalence principle, as well as the influence of an external force field on such a solution, is presented.

I. Introduction

In A. Einstein and N. Rosen "The Particle Problem in the General Theory of Relativity" one can read: "One of the imperfections of the original relativistic theory of gravitation was that as a field theory it was not complete; it introduced the independent postulate that the law of motion of a particle is given by the equation of the geodesic." [1]. This postulate states: *Free massive point particles traverse timelike geodesics.* Einstein tried to remedy that shortcoming without success. "Over the last century numerous ostensible proofs claiming to have derived the geodesic principle from Einstein's field equations have been developed. (…) Grouping these results into three major families, which I refer to as (1) limit operation proofs, (2) 0th-order proofs, and (3) singularity proofs, (…) none of these strategies successfully demonstrates the geodesic principle, canonically interpreted as a dynamical law that massive bodies must actually follow geodesic paths in Einstein's theory." [2] "By reviewing the three major classes of proof, we have seen that would-be geodesic following bodies are forced either (i) to meet unrealistically restrictive special-case conditions, (ii) to have no matter-energy at all (i.e. vanish), (iii) to violate Einstein's field equations, or (iv) to be located on paths that don't just fail to be geodesic but fail to exist in the space-time manifold at all." [2] "Though the geodesic principle can be recovered as theorem in general relativity, it is not a consequence of Einstein's equation (or the conservation principle) alone. Other assumptions are needed to drive the theorems in question." [3]. – The following is a proof of the geodesic principle and its consequences for the understanding of passive mass. The proof is not canonical in the sense that it does not directly confirm the solution but rather its sufficient convergence, which is linked to the diameter: Ø of a spatial domain that encompasses the body and the current point on the geodesic. The requirement here is that the solution bound is at most O(Ø). The proof is not based on the distributions but on density moments and can be assigned to the family of limit operation proofs. In contrast to the Geroch-Jang theorem, it is advantageous that it does not require the "strengthened dominant energy condition" [3], only the natural condition of the minimal positive body energy: $E_0 = mc^2$ in the locally inertial (LI) proper frame of reference is employed. It is also presumed that, in the vicinity of the geodesic without the gravitational effect of the body itself, the given metric field function is sufficiently smooth. Furthermore, the compatibility with the weak equivalence principle is required. The physically relevant case where the body's density is constrained ($m = O(d^3)$) is analyzed here. It is demonstrated that even for $m = O(d)$, the gravitational field that originates from the body remains sufficiently separated from the external gravitational field, allowing the test body problem to be limited to an extent that because of the adequate rate of convergence it plays a marginal role in the overall solution. The question of whether the solution converges to a geodesic at all when the mass is bounded by $O(d^0)$, which would correspond to the canonical account [2], remains open here. – In the first part: (1,2) a suitable stationary (S)LI coordinate system is constructed. In the second part: (3,4) the approximation -uncertainties, -errors and the deviation of the four-momentum derivative are estimated. Lastly, in the third part: (5) the geodesic principle is confirmed with the SE-tensor and the geodesic equation is derived from the reduced SE-tensor. For the sake of simplicity, natural units are used in the following sections. Moreover, in the summation notation, to provide a better overview, the corresponding indices are visibly crossed out when summing.

II. The physics behind the geodesic principle

1) The locally gauged, stationary locally (in the $\Delta\tau$ span) inertial coordinate system: $\underline{x}^{\hat{\mu}}: \mathcal{P} \mapsto x^{\hat{\mu}}(\mathcal{P})$

▷ a) A space-time coordinate system: $\underline{x}^{\mu}$ with its basis: $\boldsymbol{e}_{\alpha}$ and the metric: $g_{\alpha\beta}$. $\eta \equiv [\mathrm{diag}(-1,1,1,1)]$

$$\tau \in \mathbb{R}\ ;\quad \underline{x}^{\mu}:\ \forall\, \mathrm{x}_{\prime}^{\mu}(\mathcal{P}(\tau))\ \exists\, \Lambda_{\mu}^{\nu'}(x^{\mu} \to x_{\prime}^{\mu})\ ,\quad g_{\alpha\beta} \equiv \boldsymbol{e}_{\alpha}\cdot\boldsymbol{e}_{\beta} = \Lambda_{\alpha}^{\alpha'}\Lambda_{\beta}^{\beta'}\eta_{\alpha'\beta'}\ ; \qquad (1.1)$$

▷ b) For *any* $\underline{x}^{\mu}$, the *stationary locally inertial* (SLI) coordinate system: $\underline{x}^{\hat{\mu}}$ is (implicitly) pre-defined

$$x^{\tilde{0}} := \tau \qquad \to \qquad \mathcal{P}(x^{\tilde{0}},\ x^{\tilde{n}} = 0) := \mathcal{P}(\tau) \qquad (1.2a,b)$$

$$\mathbf{e}_{\tilde{\alpha}} \equiv \mathbf{e}_{\tilde{\alpha}}(\tau) := \boldsymbol{e}_{\tilde{\alpha}}(x^{\tilde{0}},\ x^{\tilde{n}} \to 0) \quad ; \quad \Lambda_{\tilde{\nu}}^{\mu} \equiv \Lambda_{\tilde{\nu}}^{\mu}(\tau) := \Lambda_{\tilde{\nu}}^{\mu}(x^{\tilde{0}},\ x^{\tilde{n}} \to 0) \qquad (1.3a,b)$$

(1.8a) (1.9a)
$$\boldsymbol{\Delta}\mathcal{P}(\tau) = \boldsymbol{e}_{\alpha}(\mathcal{P}(\tau))\Lambda_{\tilde{0}}^{\alpha}(\tau)\Delta\tau := \mathbf{e}_{\tilde{0}}(\tau)\Delta\tau \ \big|\ \Delta\tau \to 0 \quad \to \quad \frac{\eth \mathrm{x}_{\prime}^{\mu}(\mathcal{P}(\tau))}{\eth\tau} = \Lambda_{\tilde{0}}^{\mu}(\tau) \qquad (1.4a,b)$$

(2.5a) (2.5b)
$$x^{\mu} =: \mathrm{x}_{\prime}^{\mu}(\mathcal{P}(\tau)) + \Lambda_{\tilde{n}}^{\mu}(\tau)x^{\tilde{n}} + 2^{-1}\Lambda_{\tilde{n},\tilde{m}}^{\mu}(\tau)x^{\tilde{n}}x^{\tilde{m}} \ \big|\ 2|x^{\hat{k}}| \le \emptyset_{0} : \text{"small enough"} \qquad (1.5)$$

▷ c) Conditions for the SLI basis in the (infinitesimal) proximity: $x^{\tilde{n}} \to 0$; of any point: $\mathcal{P}(\tau)$ of the trajectory following the geodesic (a kind of situation like inside a freely moving non-rotating spaceship)

$$\mathrm{g}_{\tilde{\alpha}\tilde{\beta}}(\tau) := g_{\tilde{\alpha}\tilde{\beta}}(x^{\tilde{0}},\ x^{\tilde{n}} \to 0) \equiv \mathbf{e}_{\tilde{\alpha}}\cdot\mathbf{e}_{\tilde{\beta}} := \eta_{\tilde{\alpha}\tilde{\beta}} \quad \to \quad \mathbf{e}_{\tilde{0}}\cdot\mathbf{e}_{\tilde{0}} = -1 \qquad (1.6a,b)$$

(1.6b) (2.5b)
$$\mathbf{e}_{\tau}(\tau) := \mathbf{e}_{\tilde{0}}(\tau) \quad : \quad \frac{\eth\mathbf{e}_{\tau}}{\eth\tau} \equiv \frac{\partial\mathbf{e}_{\tilde{0}}}{\partial x^{\tilde{0}}} = 0 \quad \to \quad \Gamma^{\tilde{\gamma}}{}_{\tilde{0}\tilde{0}}(\tau) := \Gamma^{\tilde{\gamma}}{}_{\tilde{0}\tilde{0}}(x^{\tilde{0}},\ x^{\tilde{n}} \to 0) = 0 \qquad (1.7a,b)$$

(1.2a) (1.6a)
$$\frac{\partial\Lambda_{\tilde{0}}^{\mu}}{\partial x^{\tilde{\nu}}} \equiv \frac{\partial\Lambda_{\tilde{\nu}}^{\mu}}{\partial x^{\tilde{0}}} \equiv \frac{\eth\Lambda_{\tilde{\nu}}^{\mu}}{\eth\tau} \ :\ \frac{\partial\mathbf{e}_{\tilde{\alpha}}}{\partial x^{\tilde{0}}} \equiv \frac{\partial\mathbf{e}_{\tilde{0}}}{\partial x^{\tilde{\alpha}}} = 0 \quad \to \quad \Gamma^{\tilde{\gamma}}{}_{\tilde{\alpha}\tilde{0}}(\tau) := \Gamma^{\tilde{\gamma}}{}_{\tilde{\alpha}\tilde{0}}(x^{\tilde{0}},\ x^{\tilde{n}} \to 0) = 0 \qquad (1.8a,b)$$

$$\Lambda_{\tilde{m},\tilde{n}}^{\mu} \equiv \Lambda_{\tilde{n},\tilde{m}}^{\mu} := \frac{\partial\Lambda_{\tilde{n}}^{\mu}}{\partial x^{\tilde{m}}} \ :\ \frac{\partial\mathbf{e}_{\tilde{\alpha}}}{\partial x^{\tilde{m}}} \equiv \frac{\partial\mathbf{e}_{\tilde{m}}}{\partial x^{\tilde{\alpha}}} = 0 \quad \to \quad \Gamma^{\tilde{\gamma}}{}_{\tilde{\alpha}\tilde{\beta}}(\tau) := \Gamma^{\tilde{\gamma}}{}_{\tilde{\alpha}\tilde{\beta}}(x^{\tilde{0}},\ x^{\tilde{n}} \to 0) = 0 \qquad (1.9a,b)$$

▷ d) The SLI gauge transformation of the $\underline{x}^{\tilde{\mu}}$ SLI coordinates and the SLI Lorentz gauge as its example

$$2|x^{\hat{k}}| \le \emptyset_{0} \quad \Rightarrow \quad x^{\hat{\mu}} := x^{\tilde{\mu}} + \hat{\xi}^{\tilde{\mu}} \ \big|\ \hat{\xi}^{\tilde{\mu}}(\tau,0) = 0\ ,\ \hat{\xi}^{\tilde{\mu}}{}_{,\tilde{\nu}}(\tau,0) = 0\ ,\ \hat{\xi}^{\tilde{\mu}}{}_{,\tilde{\nu},\tilde{\kappa}}(\tau,0) = 0 \qquad (1.10)$$

[5]
$$\hat{\xi}^{\tilde{\mu}}(\tau, x^{\tilde{n}}) :\ \bar{h}^{\hat{\alpha}\hat{\nu}}{}_{,\hat{\nu}} = 0 \quad \leftarrow \quad \bar{h}^{\hat{\alpha}\hat{\beta}} := h^{\hat{\alpha}\hat{\beta}} - 2^{-1}\,\eta^{\hat{\alpha}\hat{\beta}}\eta^{\hat{\alpha}\hat{\beta}}h_{\hat{\alpha}\hat{\beta}} \quad \leftarrow \quad h_{\hat{\alpha}\hat{\beta}} := \Delta g_{\hat{\alpha}\hat{\beta}} \qquad (1.11a,b,c)$$

2) General definitions in the context of the body's stress-energy (SE-)tensor field: $T^{\mu\nu}(x^{\mu}) := T^{\nu\mu}(x^{\mu})$

▷ a) A closed spatial region: $\underline{V}(\tau) \in \underline{\boldsymbol{V}}$ of the *minimal* diameter: $\emptyset$ containing the whole body *and* $\mathcal{P}(\tau)$

$$\underline{V} := \underline{V}(\tau):\ \underline{V} \cup \partial\underline{V} = \underline{V}\ ,\ \ \mathcal{P}(x^{\hat{\mu}}) \in \underline{V}(\tau) \ \Rightarrow\ \mathcal{P}(x^{\hat{\mu}}) = \mathcal{P}(\tau, x^{\hat{n}})\ ; \qquad (2.1)$$

$$\underline{V}(\tau):\ \mathcal{P}(\tau, x^{\hat{n}}) \in (\partial\underline{V} \cup \overline{\underline{V}}) \ \Rightarrow\ T^{\alpha\beta}(\tau, x^{\hat{n}}) = 0 \qquad (2.2)$$

The body diameter:
$$\mathrm{d} := \mathrm{d}(\tau) :\le \emptyset := \emptyset(\tau) := \emptyset\left(\underline{V}(\tau)\right) :\le \frac{1}{2}\emptyset_{0} \qquad (2.3)$$

▷ b) The notation of a spatial volume integral over $\underline{V}$, which is embedded in its space-time domain: $\underline{\boldsymbol{V}}$

$$< f > := \int_{\underline{V}} f\, |\eth\underline{V}| \quad \leftarrow \quad |\eth\underline{V}| := \sqrt{-g}\ |\eth x^{1}\eth x^{2}\eth x^{3}|$$: a proper spatial volume element (2.4a,b)

▷ c) Synchronizing (initial) condition for $\underline{x}^{\hat{\mu}}(x^{\hat{0}} = \tau_{0})$, which codetermine the matrix: $\Lambda_{\tilde{\nu}}^{\mu}$ at $\tau = \tau_{0}$

(1.4,5)
$$\begin{cases} \mathcal{P}(\tau_{0})\ :\ < x^{\hat{n}}T^{\hat{0}\hat{0}}(\tau_{0}, x^{\hat{n}}) > \ = \ 0 & \leftarrow \text{ position : 1th moments of } T^{\hat{0}\hat{0}} \qquad (2.5a) \\ \mathbf{e}_{\tilde{0}}(\tau_{0}):\ < T^{\hat{n}\hat{0}}(\tau_{0}, x^{\hat{n}}) > \ = 0 & \leftarrow \text{ velocity : 0th moment of } T^{\hat{n}\hat{0}} \qquad (2.5b) \end{cases}$$

If the SLI coordinate system fulfills this condition at $\tau = \tau_{0}$, it can on $\underline{V}(\tau \to \tau_{0})$ be referred to as the (locally inertial momentarily comoving) *proper frame* (of reference) and after that, as long as $2\emptyset \le \emptyset_{0}$ is satisfied, as the *locally inertial* (LI) *comoving frame* (of reference). The parameter: τ is the proper time.

3) The locally inertial coordinates: the vector integration and the SE-tensor divergence spatial integral

▷ a) The flat basis approximation and factoring of the local coordinates basis out of the spatial integral

The norm: $|x^{\bar{n}}|$ is described as the *spatial* distance that is the shortest length from $\mathcal{P}(x^{\bar{\nu}})$ to $\mathcal{P}(x^{\bar{0}}, 0^{\bar{n}})$.

$$\mathbf{e}_{\alpha} := \boldsymbol{e}_{\alpha}\big(x^{\alpha} \to x_{,}{}^{\alpha}(\mathcal{P}_0)\big) \quad \to \quad \frac{\partial \mathbf{e}_{\alpha}}{\partial x^{\beta}} \equiv \Gamma^{\gamma}{}_{\alpha\beta}\mathbf{e}_{\gamma} \equiv \Gamma^{\gamma}{}_{\beta\alpha}\mathbf{e}_{\gamma} := \Gamma^{\gamma}{}_{\alpha\beta}\big(x^{\alpha} \to x_{,}{}^{\alpha}(\mathcal{P}_0)\big)\,\mathbf{e}_{\gamma} \qquad (3.1a,b)$$

$$\boldsymbol{e}_{\alpha} = \mathbf{e}_{\alpha} + \Gamma^{\mu}{}_{\alpha\nu}\mathbf{e}_{\mu}x^{\nu} + \tilde{O}\left(2^{-1}\left|\Gamma^{\mu}{}_{\alpha\kappa,\gamma} + \Gamma^{\beta}{}_{\alpha\kappa}\Gamma^{\mu}{}_{\beta\gamma}\right| \mathbf{e}_{\mu}|x^{\kappa}x^{\gamma}|\right) \qquad (3.2)$$

$$\underline{x}^{\mu} \;\to\; \underline{x}^{\bar{\mu}}(\mathcal{P}_0) \;:\; x^{\bar{\mu}}(\mathcal{P} = \mathcal{P}_0) = 0 \;,\; \Gamma^{\bar{\gamma}}{}_{\bar{\alpha}\bar{\beta}} = 0 \;,\; g_{\bar{\mu}\bar{\nu};\bar{\beta}} \equiv 0 \;\to\; g_{\bar{\mu}\bar{\nu},\bar{\beta}} = 0 \qquad (3.3a,b)$$

$$\boldsymbol{e}_{\bar{\alpha}} = \mathbf{e}_{\bar{\alpha}} + \tilde{O}\big(2^{-2}\big|(\Gamma^{\bar{\mu}}{}_{\bar{\alpha}\bar{\kappa},\bar{\gamma}} + \Gamma^{\bar{\mu}}{}_{\bar{\alpha}\bar{\gamma},\bar{\kappa}}) - \mathrm{R}^{\bar{\mu}}{}_{\bar{\alpha}\bar{\kappa}\bar{\gamma}}\big||x^{\kappa}||x^{\gamma}|\mathbf{e}_{\bar{\mu}}\big) = \mathbf{e}_{\bar{\alpha}} + \tilde{O}^{\bar{\mu}}{}_{\bar{\alpha}}\big(|x^{\bar{0}}|^2 + |x^{\bar{n}}|^2\big)\mathbf{e}_{\bar{\mu}} \qquad (3.4)$$

$$\big(x^{\bar{0}} \leftarrow 0\big) \;\to\; \boldsymbol{e}_{\bar{\alpha}} = \mathbf{e}_{\bar{\alpha}} + \tilde{O}^{\bar{\mu}}{}_{\bar{\alpha}}(|x^{\bar{n}}|^2)\mathbf{e}_{\bar{\mu}} \qquad (3.5)$$

$$Y^{\alpha} \!:\in \mathbb{R}^4 \;\to\; < Y^{\bar{\alpha}}\boldsymbol{e}_{\bar{\alpha}} > \,=\, < Y^{\bar{\alpha}}\mathbf{e}_{\bar{\alpha}} > + < Y^{\bar{\alpha}}\,\tilde{\mathbf{O}}_{\bar{\alpha}}(|x^{\bar{n}}|^2) > \,=\, < Y^{\bar{\alpha}} > \mathbf{e}_{\bar{\alpha}} + \big\|Y^{\bar{\alpha}}\big\|\,\tilde{\mathbf{O}}_{\bar{\alpha}}(\emptyset^2) \qquad (3.6)$$

(1.9b) (3.3) $$\forall\,\tau,\quad \underline{\underline{x^{\hat{\mu}} - x_{,}{}^{\hat{\mu}}(\mathcal{P}(\tau_{,}))}} \subset \underline{x^{\bar{\mu}}(\mathcal{P}(\tau_{,}))} \;\Big|\; 2|x^{\hat{k}}| \leq \emptyset_0 \qquad (3.7)$$

This means that (3.6) is also valid for $\underline{x}^{\hat{\mu}}$. For the rest the LI coordinates components can be separated:

(3.6) $$\forall\hat{\alpha} \qquad < Y^{\hat{\alpha}}\boldsymbol{e}_{\hat{\alpha}} > \;=\; \Big(< Y^{\hat{\alpha}} > + \tilde{O}^{\hat{\alpha}}\big(\|Y^{\hat{\mu}}\|\emptyset^2\big)\Big)\,\mathbf{e}_{\hat{\alpha}} \qquad (3.8)$$

The uncertainty: $\tilde{O}^{\hat{\alpha}}(*\,\emptyset^2)$ is the cost of making the vector integration over a curved domain reasonable.

▷ b) The total neutrality of spatial divergence for the body's SE-tensor field in the LI comoving frame

(2.2) $$T^{\hat{\alpha}\hat{\beta}}\big(\mathcal{P} \in \underline{S} := \partial\underline{V}\big) \equiv 0 \;\to\; < T^{\hat{\mu}\hat{n}}{}_{,\hat{n}}\big(\tau, x^{\hat{k}}\big) > + \tilde{O}^{\hat{\mu}}(\mathrm{m}\emptyset^2) = \oint_{\underline{S}} T^{\hat{\mu}\hat{n}}\big(\tau, x^{\hat{k}}\big)\, n_{\hat{n}}\big|\eth\underline{S}\big| \equiv 0 \qquad (3.9)$$

4) Finding the $O(\mathrm{m}\emptyset^2)$ approximation of the four-momentum temporal partial derivative for a small body

▷ a) The test body problem: The cross effect of gravity fields, its convergence upper bound estimation

Based on the equation for weak gravitational field and its source given by a hypothetical scalable massive body with a density limit (no singularities), which on closer inspection can be treated as the superposition of the infinitesimally small dispersed point masses with their $\eth\bar{h}_{\hat{0}\hat{0}} \approx 4\eth\rho/r$ partial fields; by negligible external reflections, the following basic estimates of the metric deviation and its derivatives can be found out in the originally (without the active body mass as the field source) LI comoving frame.

[5] (1.11) $$\eta^{\hat{\mu}\hat{\nu}}\,\bar{h}^{\hat{\alpha}\hat{\beta}}{}_{,\hat{\mu},\hat{\nu}} \equiv \eta^{\hat{\mu}\hat{\nu}}\left(h^{\hat{\alpha}\hat{\beta}} - \frac{1}{2}\,\eta^{\hat{\alpha}\hat{\beta}}h\right)_{,\hat{\mu},\hat{\nu}} = -16\pi T^{\hat{\alpha}\hat{\beta}} \quad \leftarrow \quad h = h^{\hat{\alpha}}{}_{\hat{\alpha}} = \eta^{\hat{\alpha}\hat{\beta}}h_{\hat{\alpha}\hat{\beta}} \qquad (4.1a,b)$$

(2.3) (5.1) $$T^{\hat{\alpha}\hat{\beta}} := O^{\hat{\alpha}\hat{\beta}}(\mathrm{d}^0) \in \mathbb{R}^{4\times4} \;\to\; \mathrm{m} := \left\|T^{\hat{\alpha}\hat{\beta}}\right\| \;\to\; \mathrm{m} := 8\mathrm{m}_0\emptyset_0^{-3}\mathrm{d}^3 = O(\mathrm{d}^3) \qquad (4.2a,b,c)$$ ℘(4.3)

$$g_{\hat{\alpha}\hat{\beta}} := \eta_{\hat{\alpha}\hat{\beta}} + \Delta g_{(ex)\hat{\alpha}\hat{\beta}} + \Delta g_{(in)\hat{\alpha}\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}} + h_{(ex)\hat{\alpha}\hat{\beta}} + h_{(in)\hat{\alpha}\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}} + O_{\hat{\alpha}\hat{\beta}}\big(\emptyset^2\big) + O_{\hat{\alpha}\hat{\beta}}(\mathrm{md}^{-1})$$

$$h_{(ex)\hat{\alpha}\hat{\beta}} = O_{\dots}(\emptyset^2) \;\to\; h_{(ex)\hat{\alpha}\hat{\beta},\hat{\gamma}} = O_{\dots}(\emptyset) \;\to\; h_{(ex)\hat{\alpha}\hat{\beta},\hat{\gamma},\hat{\varepsilon}} = O_{\dots}(\emptyset^0) \;\to\; h_{(ex)\hat{\alpha}\hat{\beta},\hat{\gamma},\hat{\varepsilon},\hat{\kappa}} = O_{\dots}(\emptyset^0) \qquad (4.4a..d)$$

$$h_{(in)\hat{\alpha}\hat{\beta}} = O_{\dots}(\mathrm{md}^{-1}) \to h_{(in)\hat{\alpha}\hat{\beta},\hat{\gamma}} = O_{\dots}(\mathrm{md}^{-2}) \to h_{(in)\hat{\alpha}\hat{\beta},\hat{\gamma},\hat{\varepsilon}} = O_{\dots}(\mathrm{md}^{-3}) \to h_{(in)\hat{\alpha}\hat{\beta},\hat{\gamma},\hat{\varepsilon},\hat{\kappa}} = O_{\dots}(\mathrm{md}^{-4}) \qquad (4.5a..d)$$

This two fields: of the body (internal) and of the externally imposed curvature, are superimposed resulting in two partial Christoffel symbol fields and creating additionally the cross-term: Δ_{X} in which range the separation between the two fields is no longer definite. The convergence order of this term and its derivatives can be determined. As Newton's first law applies, the internal field itself in an inertial system can have no effect on the overall motion, hence it is arbitrarily omitted but the cross-term remains valid.

$$\Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} := \Gamma_{(in)}{}^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} + \Gamma_{(ex)}{}^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} + \Delta_{\mathrm{X}}{}^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} \qquad (4.6)$$ ℘(4.7)

$$\Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} \approx \frac{\big\{\eta_{\hat{\mu}\hat{\nu}} - h_{(in)\hat{\mu}\hat{\nu}} - h_{(ex)\hat{\mu}\hat{\nu}}\big\}^{\hat{\mu}\hat{\nu}}}{2}\Big(h_{(in)\hat{\nu}\hat{\alpha},\hat{\beta}} + h_{(in)\hat{\nu}\hat{\beta},\hat{\alpha}} - h_{(in)\hat{\alpha}\hat{\beta},\hat{\nu}} + h_{(ex)\hat{\nu}\hat{\alpha},\hat{\beta}} + h_{(ex)\hat{\nu}\hat{\beta},\hat{\alpha}} - h_{(ex)\hat{\alpha}\hat{\beta},\hat{\nu}}\Big)$$

$$\Delta_{X}{}^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} \approx \frac{\{h_{(in)\hat{\mu}\hat{\nu}}\}^{\hat{\mu}\hat{\nu}}}{2}\left(h_{(ex)\hat{\alpha}\hat{\beta},\hat{\nu}} - h_{(ex)\hat{\nu}\hat{\alpha},\hat{\beta}} - h_{(ex)\hat{\nu}\hat{\beta},\hat{\alpha}}\right) + \frac{\{h_{(ex)\hat{\mu}\hat{\nu}}\}^{\hat{\mu}\hat{\nu}}}{2}\left(h_{(in)\hat{\alpha}\hat{\beta},\hat{\nu}} - h_{(in)\hat{\nu}\hat{\alpha},\hat{\beta}} - h_{(in)\hat{\nu}\hat{\beta},\hat{\alpha}}\right)$$

(4.4,5,6) (5.3a) $$\left(\Gamma_{(in)}{}^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} \leftarrow 0 \ , \ \tau \to \tau_0\right) \ \Rightarrow \ \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} = \Gamma_{(ex)}{}^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} + \Delta_{X}{}^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} = \Gamma_{(ex)}{}^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} + O^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}}(\mathrm{md}^0) \quad (4.8)\ (4.9a,b)$$

$$\to \quad \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta},\hat{\gamma}} = \Gamma_{(ex)}{}^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta},\hat{\gamma}} + O^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}\hat{\gamma}}(\mathrm{md}^{-1}) \quad \to \quad \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta},\hat{\gamma},\hat{\varepsilon}} = \Gamma_{(ex)}{}^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta},\hat{\gamma},\hat{\varepsilon}} + O^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\varepsilon}}(\mathrm{md}^{-2}) \quad (4.9c,d)$$

Based on (4.9), it can be shown that in (4.19) the resulting cross-connection error would converge one order faster than the approximation error there. Additionally, it is worth noting that even for m = O(d) the solutions (5.8,21) still converge at O(mØ) rate. Hence, the cross-term: Δ_X is neglected from here on.

▷ b) The approximate factoring of the Christoffel symbol out of the spatial integral over the volume: $\underline{V}$

(1.6a) $$\Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}}(\tau, x^{\hat{n}}) = \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}}(g_{\hat{*}\hat{*}}, g_{\hat{*}\hat{*},\hat{*}}) + \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta},\hat{n}}x^{\hat{n}} + 2^{-1}O\left(\left|\Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta},\hat{k},\hat{l}}\right| |x^{\hat{k}}x^{\hat{l}}|\right) := \Gamma_{(ex)}{}^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} \quad (4.10)$$

$$< \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}}T^{\hat{\gamma}\hat{\nu}} > = \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} < T^{\hat{\gamma}\hat{\nu}} > + \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta},\hat{n}} < x^{\hat{n}}T^{\hat{\gamma}\hat{\nu}} > + 0^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}}\left(\|T^{\hat{\gamma}\hat{\nu}}\|Ø^2\right) \quad (4.11)$$

$$\mathrm{T}^{\hat{\gamma}\hat{\nu}} := < T^{\hat{\gamma}\hat{\nu}} > \quad \to \quad < \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}}T^{\hat{\gamma}\hat{\nu}} > = \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}}\mathrm{T}^{\hat{\gamma}\hat{\nu}} + \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta},\hat{n}}\mathrm{T}^{\hat{\gamma}\hat{\nu}\langle\hat{n}\rangle} + 0^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}}\left(\|T^{\hat{\gamma}\hat{\nu}}\|Ø^2\right) \quad (4.12a,b)$$

With (1.9b) it leads to the upper bound estimation of deviation of the temporal partial derivative (4.19).

(4.11) (1.9b) $$< \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}}T^{\hat{\gamma}\hat{\nu}} > = 0\left(\left|\Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta},\hat{n}}\right| \|T^{\hat{\gamma}\hat{\nu}}\|Ø\right) + 0^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}}\left(\|T^{\hat{\gamma}\hat{\nu}}\|Ø^2\right) = 0^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}}\left(\|T^{\hat{\gamma}\hat{\nu}}\|Ø\right) \quad (4.13)$$

(4.2) $$< \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}}T^{\hat{\alpha}\hat{\beta}} > = 0^{\hat{\mu}}(\mathrm{m}Ø) \quad ; \quad < \Gamma^{\hat{\beta}}{}_{\hat{\alpha}\hat{\beta}}T^{\hat{\mu}\hat{\alpha}} > = 0^{\hat{\mu}}(\mathrm{m}Ø) \quad (4.14a,b)$$

▷ c) The temporal partial derivative of the four-momentum, obtained from the conservation condition:

$$T^{\mu\beta}{}_{;\beta} \equiv T^{\mu\beta}{}_{,\beta} + \Gamma^{\mu}{}_{\alpha\beta}T^{\alpha\beta} + \Gamma^{\beta}{}_{\alpha\beta}T^{\mu\alpha} := 0 \quad (4.15)$$

$$T^{\mu\beta}{}_{,\beta} = -\Gamma^{\mu}{}_{\alpha\beta}T^{\alpha\beta} - \Gamma^{\beta}{}_{\alpha\beta}T^{\mu\alpha} \quad (4.16)$$

(3.8) $$< T^{\hat{\mu}\hat{\beta}}{}_{,\hat{\beta}} > = - < \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}}T^{\hat{\alpha}\hat{\beta}} > - < \Gamma^{\hat{\beta}}{}_{\hat{\alpha}\hat{\beta}}T^{\hat{\mu}\hat{\alpha}} > + \tilde{0}^{\hat{\mu}}(\mathrm{m}Ø^2) \quad (4.17)$$

(3.9) $$< T^{\hat{\mu}\hat{0}}{}_{,\hat{0}} > = - < \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}}T^{\hat{\alpha}\hat{\beta}} > - < \Gamma^{\hat{\beta}}{}_{\hat{\alpha}\hat{\beta}}T^{\hat{\mu}\hat{\alpha}} > + \tilde{0}^{\hat{\mu}}(\mathrm{m}Ø^2) \quad (4.18)$$

(4.11,14) (1.9b) $$< T^{\hat{\mu}\hat{0}}{}_{,\hat{0}} > = -\Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta},\hat{n}} < x^{\hat{n}}T^{\hat{\alpha}\hat{\beta}} > - \Gamma^{\hat{\beta}}{}_{\hat{\alpha}\hat{\beta},\hat{n}} < x^{\hat{n}}T^{\hat{\mu}\hat{\alpha}} > + 0^{\hat{\mu}}(\mathrm{m}Ø^2) = 0^{\hat{\mu}}(\mathrm{m}Ø) \quad (4.19)$$

5) The freely falling small body and its, founded on the conservation conditions, near geodesic solutions

Because $\mathcal{P}(\tau_0)$ can be any given point on the geodesic, in the ongoing section is assumed that the body is currently situated in the *proper* or at least in the LI *comoving* frame of reference, the behavior of the body in the vicinity of the spatial coordinates origin: $\mathcal{P}(\tau)$ is analyzed, and if the result follows the geodesic in the limit case (for $\mathrm{d} \to 0$), it must also follow it inside the $Ø_0$ tube for $\mathrm{d} > 0$ in a certain timespan.

▷ a) The proper: (rest) mass: m , four-position: $\mathrm{x}^{\hat{\alpha}}$, four-velocity: $\mathrm{U}^{\hat{\alpha}}$ and its minimal or rest energy: E_0

(3.8) $$\mathrm{m}(\tau = \tau_0) := < T^{\hat{0}\hat{0}}(\tau_0, x^{\hat{n}}) > + \tilde{0}(\mathrm{md}^2) \quad (5.1)$$

(2.3) $$\mathrm{x}^{\hat{0}}(\tau) := \tau \ ; \quad \mathrm{x}^{\hat{n}}(\tau) := < T^{\hat{0}\hat{0}}(\tau, x^{\hat{n}}) >^{-1} < x^{\hat{n}}T^{\hat{0}\hat{0}}(\tau, x^{\hat{n}}) > + \tilde{0}^{\hat{n}}(Ø^2\mathrm{d}) \quad (5.2a,b)$$

(2.5a) (1.2a) $$Ø(\tau_0) \equiv \mathrm{d}(\tau_0) \quad \to \quad \mathrm{x}^{\hat{0}}(\tau_0) = \tau_0 \ , \ \mathrm{x}^{\hat{n}}(\tau_0) = \tilde{0}^{\hat{n}}(\mathrm{d}^3) \quad (5.3a,b)$$

$$\mathrm{U}^{\hat{\mu}} := \frac{\mathrm{d}\mathrm{x}^{\hat{\mu}}}{\mathrm{d}\tau} \equiv \frac{\partial \mathrm{x}^{\hat{\mu}}}{\partial x^{\hat{0}}} \ \Bigg| \ |\mathrm{U}^{\hat{n}}| \ll 1 \ \leftarrow \ 0^{\hat{\alpha}}{}_{,\tau}(\mathrm{d}^k) = 0^{\hat{\alpha}}(\mathrm{d}^k) \ \Leftarrow \ \frac{0^{\hat{\alpha}}(\mathrm{d}^k)}{\mathrm{d}^k} = 0^{\hat{\alpha}}(1, \tau \to \tau_0) \in C^2 \quad (5.4a..c)$$

(2.5b) (5.3a) $$\mathrm{E}_0 := \mathrm{E}(\tau_0) = \min\left(\mathrm{m}(\tau_0)/\sqrt{1 - \mathrm{U}^{\hat{n}}(\tau_0)\mathrm{U}_{\hat{n}}(\tau_0)}\right) \ \Rightarrow \ \mathrm{U}^{\hat{n}}(\tau_0) = \tilde{0}^{\hat{n}}(\mathrm{d}^3) \quad (5.5a,b)$$

(5.2a,4) $$\mathrm{U}^{\hat{0}}(\tau_0) = 1 \ , \ \mathrm{U}^{\hat{n}}(\tau_0) = \tilde{0}^{\hat{n}}(\mathrm{d}^3) \quad \to \quad \mathrm{U}^{\hat{0}}{}_{,\hat{0}}(\tau_0) = 0 \quad (5.6a,b)$$

▷ b) The solution based on the stress-energy tensor, in the locally inertial comoving frame of reference
The four momentum: $\mathrm{p}^{\bar{\mu}}$ can be defined as the "T4-momentum": $< T^{\bar{\mu}0} >$ or as the four-velocity based "U4-momentum": $\mathrm{m}\mathrm{U}^{\bar{\mu}}$. Given they are equivalent and the mass does not vary, it follows for $|\mathrm{U}^{\hat{n}}| \ll 1$:

(5.2b) (4.19) $$\frac{\eth \mathrm{p}^{\hat{\mu}}}{\eth\tau} = -\Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta},\hat{n}}\mathrm{p}^{\hat{\alpha}}\mathrm{U}^{\hat{\beta}}\mathrm{x}^{\hat{n}} - \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta},\hat{n}} < x^{\hat{n}}T^{\hat{\alpha}\hat{\beta}} > - \Gamma^{\hat{\beta}}{}_{\hat{\alpha}\hat{\beta},\hat{n}} < x^{\hat{n}}T^{\hat{\mu}\hat{\alpha}} > + \tilde{\mathrm{O}}^{\hat{\mu}}(\mathrm{m}\emptyset^2) \quad (5.7)$$

(4.12) (4.14) $$\frac{\eth \mathrm{p}^{\hat{\mu}}}{\eth\tau} = \mathrm{O}^{\hat{\mu}}(\mathrm{m}|\mathrm{x}^{\hat{n}}|) - \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}}\mathrm{T}^{\hat{\alpha}\hat{\beta}} - \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta},\hat{n}}\mathrm{T}^{\hat{\alpha}\hat{\beta}\langle\hat{n}\rangle} - \Gamma^{\hat{\beta}}{}_{\hat{\alpha}\hat{\beta}}\mathrm{T}^{\hat{\mu}\hat{\alpha}} - \Gamma^{\hat{\beta}}{}_{\hat{\alpha}\hat{\beta},\hat{n}}\mathrm{T}^{\hat{\mu}\hat{\alpha}\langle\hat{n}\rangle} = \mathrm{O}^{\hat{\mu}}(\mathrm{m}\emptyset) \quad (5.8)$$

(5.1) (5.4a) $$\mathrm{p}^{\hat{\mu}} \approx \mathrm{m}\mathrm{U}^{\hat{\mu}} \quad \Rightarrow \quad \frac{\eth \mathrm{U}^{\hat{\mu}}}{\eth\tau} \approx \frac{\eth(\mathrm{p}^{\hat{\mu}}/\mathrm{m})}{\eth\tau} = \frac{1}{\mathrm{m}}\left(\frac{\eth \mathrm{p}^{\hat{\mu}}}{\eth\tau} - \frac{\mathrm{p}^{\hat{\mu}}}{\mathrm{m}}\frac{\eth \mathrm{m}}{\eth\tau}\right) = \mathrm{O}^{\hat{\mu}}(\emptyset) \quad (5.9a)\,(5.9b)$$

Because the origin of $\underline{x}^{\hat{n}}$ follows the geodesic, this implication proves in the limit that the body follows the geodesic too. The tidal forces in (5.7) depend also on the SLI gauge and can influence the trajectory. The critical physical problem is that (5.8) is not coordinate-invariant, and the particular thing about this is that the body must freely levitate in the $\underline{V}$ domain proximal to the spatial origin and the LI comoving frame makes it possible as gravity almost disappears there. The reason for this effect are the local translation symmetries: (t-xyz) on $\underline{V}$ in $\underline{x}^{\hat{\mu}}$, however these symmetries are not perfect since on $\underline{V}$ the derivative: $g_{\hat{\mu}\hat{\nu},\hat{k}} = O_{\hat{\mu}\hat{\nu}\hat{k}}(\emptyset) \neq 0$ and this in this case is the limiting factor for the convergence of the solution.

▷ c) The body's mass density, the object-energy (OE-)tensor (field) and its *local integral* divergence
The state of body's overall motion is defined by its 4-velocity, thus to find the tensor equation of motion the SE-tensor component that incorporates only this 4-velocity is to be used and this for a small body is the OE-tensor $\boldsymbol{T}_{\mathrm{E}}$ (5.11a) reflecting the convective flux of its matter energy; however, as simultaneity in the space-time domain: $\underline{\boldsymbol{V}}$ is relative, $\boldsymbol{T}_{\mathrm{E}}$ is ambiguous depending on the SLI-$\hat{\xi}^0$ gauge in the $O(\mathrm{d}^2)$ range. Accordingly, in order to evade the problem resulting from (5.8), instead of the SE- the OE-tensor is used causing that gravitation acts exclusively upon this stress-free component. Consequently, as a result of the energy conservation due to the τ-translation symmetry (5.14a) and since for $\tau \to \tau_0$ the first moment of $\boldsymbol{T}_{\mathrm{E}}$ is nullified because of the condition (2.5a), the LHS of (5.14b) disappears creating for the $\boldsymbol{T}_{\mathrm{E}}$ field an integral covariant conservation condition, which has $\tilde{\mathrm{O}}(\mathrm{md}^2)$ convergence order in the proper frame.
- *The following equations are studied for* $\tau \to \tau_0$ and there due to (2.5a) the offset: $\mathrm{x}^{\hat{n}}$ (5.2b) is negligible.

(3.3) (3.7) $$\underline{x}^{\bar{\alpha}}(\mathcal{P}(\tau_0)): \; x^{\bar{0}} = x^{\hat{0}} - \tau_0 \;\; , \; \Lambda^{\hat{0}}_{\bar{\mu}} = \delta^0_{\bar{\mu}} \;\; , \; g_{\bar{0}\bar{n}} = 0 \quad \to \quad U^{\bar{\mu}} := \delta^{\bar{\mu}}_0 \;\; , \; \rho := T^{\bar{0}\bar{0}} \quad (5.10\text{ a,b})$$

(5.6a) (1.9b) $$T_{\mathrm{E}}^{\hat{\mu}\hat{\nu}} := \rho\, U^{\hat{\mu}}U^{\hat{\nu}} \quad \leftarrow \quad U^{\hat{\nu}} = \Lambda^{\hat{\nu}}_{\bar{\kappa}}\,\delta^{\bar{\kappa}}_0 \approx \left(\mathrm{U}^{\hat{\nu}}(\tau) - \tilde{\mathrm{O}}^{\hat{n}}(\mathrm{d}^3,\tau)\right) + \Lambda^{\hat{n}}_{\bar{0},\hat{k},\hat{\ell}}(\tau)x^{\hat{k}}x^{\hat{\ell}} \quad (5.11\text{ a,b})$$

$$T_{\mathrm{S}}^{\mu\nu} := T^{\mu\nu} - T_{\mathrm{E}}^{\mu\nu} \quad \to \quad T_{\mathrm{E}}^{\bar{\mu}\bar{\nu}}T_{\mathrm{S}\bar{\mu}\bar{\nu}} = 0 \quad \to \quad T_{\mathrm{E}}^{\mu\nu}T_{\mathrm{S}\mu\nu} \equiv 0 \quad \to \quad T_{\mathrm{E}} = -\rho \;\; , \; T_{\mathrm{S}} = 3p \quad (5.12\text{ a..d})$$

(3.8,9) (1.6a,9b) $$< T_{\mathrm{E}\;\,;\hat{\beta}}^{\hat{\mu}\hat{\beta}} > = < T_{\mathrm{E}\;\,,\hat{0}}^{\hat{\mu}\hat{0}} > + < \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}}T_{\mathrm{E}}^{\hat{\alpha}\hat{\beta}} > + < \Gamma^{\hat{\beta}}{}_{\hat{\alpha}\hat{\beta}}T_{\mathrm{E}}^{\hat{\mu}\hat{\alpha}} > + \tilde{\mathrm{O}}^{\hat{\mu}}(\mathrm{md}^2) \quad (5.13)$$

(3.3b) (2.5a) $$g_{\hat{\mu}\hat{\nu},\tau} = \tilde{O}_{\hat{\mu}\hat{\nu}}(\mathrm{d}^2) \quad \to \quad < T_{\mathrm{E}\;\,,\hat{0}}^{\hat{\mu}\hat{0}} > + < \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}}T_{\mathrm{E}}^{\hat{\alpha}\hat{\beta}} > + < \Gamma^{\hat{\beta}}{}_{\hat{\alpha}\hat{\beta}}T_{\mathrm{E}}^{\hat{\mu}\hat{\alpha}} > = \tilde{\mathrm{O}}^{\hat{\mu}}(\mathrm{md}^2) \quad (5.14\text{a,b})$$

▷ d) The coordinate-invariant solution grounded in the OE-tensor in the (LI) proper frame of reference
As seen above, (5.14) corresponds directly to (4.18) and since the body's four-position: (5.2) on the world line is defined in the same way for the OE-tensor as it is for the SE-tensor, the subsequent proof of the geodesic solution for the OE-tensor in the limit case, confirms the result (5.9b) for the SE-tensor as well.

(5.14a)(2.4) (4.11)(2.5a) $$< T_{\mathrm{E}}^{\hat{\mu}\hat{0}} >_{,\hat{0}} = -\Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}} < T_{\mathrm{E}}^{\hat{\alpha}\hat{\beta}} > - \Gamma^{\hat{\beta}}{}_{\hat{\alpha}\hat{\beta}} < T_{\mathrm{E}}^{\hat{\mu}\hat{\alpha}} > + \mathrm{O}^{\hat{\mu}}(\mathrm{md}^2) \quad (5.15)$$

(5.11,4b) $$< \rho\mathrm{U}^{\hat{\mu}}\mathrm{U}^{\hat{0}} >_{,\hat{0}} = - < \rho > \Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}}\mathrm{U}^{\hat{\alpha}}\mathrm{U}^{\hat{\beta}} - < \rho > \Gamma^{\hat{\beta}}{}_{\hat{\alpha}\hat{\beta}}\mathrm{U}^{\hat{\mu}}\mathrm{U}^{\hat{\alpha}} + \mathrm{O}^{\hat{\mu}}(\mathrm{md}^2) \quad (5.16)$$

(5.1) (5.6b) $$(\mathrm{m}\mathrm{U}^{\hat{\mu}}\mathrm{U}^{\hat{0}})_{,\hat{0}} = \mathrm{m}\mathrm{U}^{\hat{0}}\mathrm{U}^{\hat{\mu}}{}_{,\hat{0}} + \mathrm{U}^{\hat{\mu}}\mathrm{U}^{\hat{0}}\mathrm{m}_{,\hat{0}} = -\mathrm{m}\Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}}\mathrm{U}^{\hat{\alpha}}\mathrm{U}^{\hat{\beta}} - \mathrm{m}\Gamma^{\hat{\beta}}{}_{\hat{\alpha}\hat{\beta}}\mathrm{U}^{\hat{\mu}}\mathrm{U}^{\hat{\alpha}} + \mathrm{O}^{\hat{\mu}}(\mathrm{md}^2) \quad (5.17)$$

Even though here $\Gamma^{\hat{\alpha}}{}_{\hat{\nu}\hat{\kappa}} = 0$, it is the $\Gamma^{\hat{\alpha}}{}_{\hat{\nu}\hat{\kappa}}$ that carries the key information about the origin of this zero. Since $\mathrm{U}^{\hat{0}}$ is a constant value and $\mathrm{U}^{\hat{n}} \to 0$, (5.17) can be decomposed into the system of two equations:

(5.6a) (5.20b) $$\begin{cases} \mathrm{m}\mathrm{U}^{\hat{\mu}}{}_{,\hat{0}} = -\mathrm{m}\Gamma^{\hat{\mu}}{}_{\hat{\alpha}\hat{\beta}}\mathrm{U}^{\hat{\alpha}}\mathrm{U}^{\hat{\beta}} + \mathrm{O}^{\hat{\mu}}(\mathrm{md}^2) \;\; \big|\, \mathrm{O}^{\hat{0}} = 0 & (5.18a) \\ \mathrm{U}^{\hat{\mu}}\mathrm{m}_{,\hat{0}} = -\mathrm{m}\Gamma^{\hat{\beta}}{}_{\hat{\alpha}\hat{\beta}}\mathrm{U}^{\hat{\mu}}\mathrm{U}^{\hat{\alpha}} + \mathrm{O}^{\hat{\mu}}(\mathrm{md}^2) \;\; \big|\, \mathrm{O}^{\hat{n}} = \mathrm{O}^{\hat{n}}(\mathrm{md}^5) & (5.18b) \end{cases}$$

(5.1,11) $\quad -\mathrm{m}U^{\hat\mu}U^{\hat\nu} + \tilde{O}(\mathrm{md}^2) = \mathrm{m}^{\hat\mu\hat\nu} := -< T_{\mathrm{E}}^{\hat\mu\hat\nu} > \quad \rightarrow \quad \mathrm{m} \equiv \mathrm{m}^{\mu}{}_{\mu} = \eta_{\hat\mu\hat\nu}\,\mathrm{m}^{\hat\mu\hat\nu}$ (5.19a,b)

(5.3b)
(1.2a)
$$\tau \rightarrow \tau_0 \quad \begin{cases} \mathrm{m}\dfrac{\mathrm{d}U^{\hat\mu}}{\mathrm{d}\tau} = -\mathrm{m}U^{\hat\alpha}U^{\hat\beta}\Gamma^{\hat\mu}{}_{\hat\alpha\hat\beta} + O^{\hat\mu}(\mathrm{md}^2) - \mathrm{m}\Gamma^{\hat\mu}{}_{\hat\alpha\hat\beta,\hat\kappa}U^{\hat\alpha}U^{\hat\beta}\tilde{O}^{\hat\kappa}(\mathrm{d}^3) & (5.20\mathrm{a}) \\ \dfrac{\mathrm{d}\mathrm{m}}{\mathrm{d}\tau} = -\mathrm{m}\Gamma^{\hat\beta}{}_{\hat0\hat\beta}U^{\hat0} + O^{\hat0}(\mathrm{md}^2) = O(\mathrm{md}^2) & (5.20\mathrm{b}) \end{cases}$$

The neg. OE-tensor's 0th moment: (5.19), which invariant trace equals the body mass, is the mass tensor. The equations turn out to be coordinate-invariant since the first one is the four-momentum form of the geodesic equation and the second one is a scalar equation, which is always coordinate-invariant. Hence:

(5.9a)
(1.8b)
$$\forall\tau \quad \begin{cases} \mathrm{m}\dot{U}^{\mu} := \mathrm{m}\dfrac{\mathrm{d}U^{\mu}}{\mathrm{d}\tau} = -\mathrm{m}U^{\alpha}U^{\beta}\Gamma^{\mu}{}_{\alpha\beta} + O^{\mu}(\mathrm{md}^2) & \textbf{(5.21a)} \\ \quad \rightarrow \quad \dfrac{\mathrm{D}U^{\mu}}{\mathrm{d}\tau} = O^{\mu}(\mathrm{d}^2) & \textbf{(5.21b)} \\ \dot{\mathrm{m}} := \dfrac{\mathrm{d}\mathrm{m}}{\mathrm{d}\tau} = O(\mathrm{md}^2) & (5.21\mathrm{c}) \end{cases}$$

▷ e) The limit case turns out to be the (rest) mass conservation law and the standard geodesic equation:

(5.4a) $$\lambda := a\tau + b\,, \quad \mathrm{d} \rightarrow 0 \quad \Rightarrow \quad \frac{\mathrm{d}^2x^{\mu}}{\mathrm{d}\lambda^2} = -\Gamma^{\mu}{}_{\alpha\beta}\frac{\mathrm{d}x^{\alpha}}{\mathrm{d}\lambda}\frac{\mathrm{d}x^{\beta}}{\mathrm{d}\lambda} \quad ■ \qquad (5.22)$$

III. Summary

The behavior of a massive body located in the gravity field, free of other influences and with a *negligible radiation*, has been analyzed here. The body is defined on the basis of its stress-energy (SE-)tensor field, which together with the geodesic determines the space-time domain: $\underline{\boldsymbol{V}}$ on which it exists. Consequently, using the general coordinates, a stationary locally inertial (SLI) coordinate system has been constructed within the area of validity. The SLI coordinates allow, at least for a certain amount time, to conveniently describe the movement of the body. If their spatial origin matches the comoving body at a certain τ, they are referred to there as the proper frame. This configuration has been achieved by selecting the suitable initial conditions for the SLI coordinate system itself. This proper frame is locally inertial (LI) in the neighborhood of its origin. The spatial origin of the SLI coordinate system follows a geodesic, forming a kind of geodesic tunnel in space-time, nevertheless this by itself has no influence on the tensor solution. The requested SLI gauge aims to limit the fictitious tidal forces especially on the flat domain, but it can be replaced if necessary without violating the convergence order estimation, with any other gauge that complies with the boundary conditions at the spatial origin. – Furthermore, the local integral divergence of the SE-tensor over the spatial domain is essential here. If the body is sufficiently small, after flattening of the coordinate system basis, four separate conservation equations of energy and momentum arise in the LI comoving frame from the SE-tensor zero-divergence because of the translation symmetries, which in curved space are only locally possible using approximation. Therefore, it is important here to be able to estimate the convergence rate of occurring uncertainties deviations and errors. To determine the upper bound estimates the big *O*-notation, which denotes *a member* of the function class that implements an explicitly specified convergence order, has been employed. The limited spatial extent of the SE-tensor field, makes it attainable to reduce its local integral divergence to the temporal derivative. This is crucial because it makes possible to derive the body's equation of motion from the zero-divergence of the SE-tensor. To guarantee that the test body problem is not critical here, the cross effect of gravity fields was shown to be negligible up to m = O(d). – Based on the SE-tensor the body's geodesic trajectory has been found only in the SLI coordinates. This reinforces the idea that the SE-tensor as a basis for the geodesic equation, which is a tensor equation, is suitable only to a limited extent. To solve this problem, the SE-tensor has been reduced by setting all its subcomponents except the energy density *in the orthochronous proper frame* arbitrarily at zero, thus defining its component as the OE-tensor that depends there on the energy density but not on the body stress density field as seen in terms of the internal energy flux, shear stress and pressure, which form the object-stress tensor $\boldsymbol{T}_{\mathrm{S}}$. The negative 0th moment of the OE-tensor: the mass tensor (5.19) is along with mass and four-momentum, just another quantity associated with an object. Clearly, replacing the SE-tensor with its OE-tensor eliminates the dependence on the body stress. This is directly visible only for the body as the internal observer in its proper frame; an external observer perceives only the final tensor result. Now, the energy flux balance equation is solved by splitting it into two separate equations: the first one relates to the body's acceleration, the second one to its mass change. Ultimately, these equations can be represented by two tensor equations, which in the limit case become the geodesic equation and the mass conservation law for a freely falling body in the general space-time.

IV. Conclusions

For a small body diameter: d there are two verifiable possibilities: the effective gravitational tidal forces A: *can* influence the trajectory of the freely falling body at the O(d) deviation level. B: *cannot* because the internal stress tensor field of the body vanishes or on average does not interact with the gravity field. – It has been proven that even if no coordinate-invariant equation of motion based on the SE-tensor has been found, a free "point" body associated with its SE-tensor traverses timelike geodesic. Essentially, the covariant conservation condition of the SE-tensor is sufficient to determine this, provided the body is isolated from all conventional forces and, according to the local symmetries in the LI comoving frame of the SLI system, is additionally apparently isolated from gravity. In this case, the possibility "*A*" arises if the distribution of any component of the SE-Tensor does not correspond to the body's energy density. – Simultaneously this statement can be extended analogously to a solution, which involves the object-energy (OE-)tensor, which is the stress-free component of the SE-tensor, but in this case the direct coordinate-invariant solution is the geodesic equation; yet should "*A*" be true, a body's trajectory error at the O(d) level could occur. Thereby, the OE-tensor provides a mechanism through which gravity influences the massive body, while ensuring the conservation of the body's mass. By applying the SE-tensor instead of the OE-tensor in a tensor equation of motion the weak equivalence principle may be disregarded, but since there are good reasons to conclude that the weak equivalence principle remains valid, the solution on the OE-tensor basis is preferable to rule out the theoretic dependence on the stress tensor of the body's matter. - Accordingly, the following thesis should be proposed: The total gravitational influence on a sufficiently small freely falling massive body or particle is equal to this influence on its OE-tensor field. Formulated in such a way, the above thesis in the near limit case leads not only to the geodesic principle and the weak equivalence principle but also to the mass conservation law for an isolated body and to the explicit consequence that the rest mass and the passive mass must always be equal. Furthermore, this approach has the additional advantage of not being a quasi-mathematical axiom but instead having the form based on the local integral conservation condition for the body's OE-tensor field in its proper frame.

For now, there is an opportunity to substantiate the weak equivalence principle for extended objects in a (locally) uniform gravitational field. The reason is that space-time is flat there and the above derivation of the geodesic principle is exact even for such objects, because the integration uncertainty vanishes in flat space. Owing to the translation symmetry in the inertial system, the internal forces in the object are incapable to accelerate it as a whole, revealing the masking effect on the object-stress tensor in flat space. Thus, only the object (four-)velocity determines its world line path in this setup. *However*, it is important to consider that the position of such an object is defined in its proper frame; and as an integral parameter it can be unequivocal only in this context. So, the outcome is always the trajectory of the *proper* position. Because the influence of the internal forces on the object matter can be described through the covariant derivatives, the above does follow the covariant conservation condition for the object SE-tensor field. - Therefore, under these circumstances, a short-term solution expressed as the specific geodesic trajectory within a flat space-time domain, complies with the criterion defined by the weak equivalence principle.

Since the geodesic principle has not been derived here from the geometrical approach but from the conservation condition, it is natural not to limit oneself to freely falling bodies, and the external influences, such as that of the Lorentz force, can be taken into account to obtain an equation of motion like this one:

(5.9a,21a,19) $$\mathrm{m}\dot{\mathrm{U}}^{\alpha} = -\mathrm{m}\mathrm{U}^{\mu}\mathrm{U}^{\nu}\Gamma^{\alpha}_{\ \mu\nu} + \mathrm{q}\mathrm{U}^{\mu}\mathrm{F}^{\alpha}_{\ \mu} = \mathrm{m}^{\mu\nu}\Gamma^{\alpha}_{\ \mu\nu} + \mathrm{q}\mathrm{U}^{\mu}\mathrm{F}^{\alpha}_{\ \mu} \ \Big|\ \mathrm{d} \to 0 \qquad \text{(C.1)}$$

In the above equation, the mass: m is not just a result of "adjusting" the "geometrical" geodesic equation to the Lorentz force, but was already there in (5.21a). It is also clear that m on the LHS expresses the inertial mass and m on the RHS the passive mass having the same value. Moreover, it is noteworthy that the inertial mass "hides" in the body-bound coordinates where $\dot{\mathrm{U}}^{\alpha} = 0$ and the passive mass "hides" in the free-falling coordinates where $\Gamma^{\alpha}_{\ \mu\nu} = 0$, as in the famous free-falling elevator thought experiment. – This equation implies as well that the gravitational interaction can be reduced to a lower level and the thesis, which has been proposed above, may be generalized in the postulate: The gravitational influence upon a massive object results only from the gravity interaction with the OE-tensor fields of its particles. However, at a certain level a definition of the OE-tensor for quantum objects would be necessary here. – Since it is quite possible that at the quantum level the simultaneity within every wave-particle (domain) is absolute, the ambiguity of the particle OE-tensor fields may well vanish and the whole-object interaction with the gravity field would become unequivocally describable upon consideration of this level.

V. Symbols

a, b ; C^2; d	constant real values ; differentiability class ; object's (body's) diameter
đ , D , ∂	differential, absolute differential, partial differential
đ$\underline{S}$, đ$\underline{V}$, $\partial\underline{V}$	spatial surface element, spatial volume element, spatial volume boundary
|đ$\underline{V}$| , |đ$\underline{S}$|	Measure of the -proper spatial volume element (2.4b), -proper area element
Δ , $\boldsymbol{\Delta}$; Δ_X	difference, difference vector ; external-internal gravity cross-term (4.6)
δ^{μ}_{ν}	Kronecker delta. $\delta^{\mu}_{\nu} := 0$ for $\mu \neq \nu$, $\delta^{\mu}_{\nu} := 1$ for $\mu = \nu$
$\boldsymbol{e}_\alpha$, $\mathbf{e}_\alpha$, $\mathbf{e}_\alpha$	coordinate basis, coordinate basis vector, coordinate basis near/at the (spatial) origin
$\eta_{\alpha\beta}, \eta^{\alpha\beta}$	Minkowski metric: $\equiv [\mathrm{diag}(-1,1,1,1)]_{\alpha\beta}$, $\equiv [\mathrm{diag}(-1,1,1,1)]^{\alpha\beta}$;
$F_{\alpha\beta}$; Ø , Ø$_0$	electromagnetic tensor ; diameter of the spatial -domain: $\underline{V}$, -validity domain: $\underline{V}_0$
$g_{\alpha\beta}$, $\mathrm{g}_{\alpha\beta}$, g	metric field , metric near/at the (spatial) origin ; $\det([g_{\alpha\beta}])$
$\Gamma^{\alpha}{}_{\mu\nu}$, $\Gamma^{\alpha}{}_{\mu\nu}$	Christoffel symbol (field), Christoffel symbol near/at the (spatial) origin
$h_{\alpha\beta}; h^{\alpha\beta}, \bar{h}^{\alpha\beta}$	perturbation of the Minkowski metric; $h^{\alpha\beta} := \eta^{\alpha\mu}\eta^{\beta\nu}h_{\mu\nu}$, its trace reverse
$\Lambda^{\kappa'}_{\nu}, \Lambda^{\kappa'}_{\nu}$	coordinates transformation matrix, -near/at the (spatial) origin
m , $\mathrm{m}^{\mu\nu}$; n^n	(object's) -mass, -mass tensor ; (unit) normal (contra)vector
$O, \tilde{O}$; $\mathrm{O}, \tilde{\mathrm{O}}$	big-*O* symbol (converges to 0), -with pure integration uncertainty ; same τ dependent
$\mathcal{P}(x^\nu)$, $\mathcal{P}_0$	event in (or point of) the space-time, -at the coordinate system origin
p ; p^μ ; q	pressure (field) ; (object's) -four-momentum ; -electric charge
$\mathrm{R}^{\alpha}{}_{\mu\nu\sigma}$; r ; ρ	Riemann curvature tensor near/at the origin ; radius-distance $\in \mathbb{R}_+$; energy density
$T^{\mu\nu}, T^{\mu\nu}_{\mathrm{E}}, T^{\mu\nu}_{S}$	fields: stress-energy (SE-)tensor, object-energy (OE-)tensor, object-stress (OS-)tensor
$\mathrm{T}^{\mu\nu}, \mathrm{T}^{\mu\nu\langle n\rangle}$	(proper) 0th (4.12a) and the 1th (4.11,12b), spatial moments of a tensor field
τ, τ_0	proper time , proper time initial value (for the proper *and* comoving frame)
U^μ, U^μ	four-velocity field, (object's) four-velocity; ((5.11b): $U^{\hat{\nu}} = Y^{\hat{n}} \Rightarrow U^{\hat{0}} = 0$)
$\underline{V}$, $\underline{V}_0$	the minimal diameter spatial domain with the body and the current trajectory point, the spherical validity domain with the center at the current trajectory point
$\overline{\underline{V}}$, $\underline{\boldsymbol{V}}$	all space but $\underline{V}$, minimal space-time domain containing the whole $\underline{V}$ for all valid τ .
$x^\alpha, x^\alpha, \underline{x}^\alpha, \underline{\underline{x}}^\alpha$	coordinate, coordinates, (all) coordinates = coordinate -system, -systems (their full set)
$x^\alpha, x^{\bar{\alpha}}, x^{\tilde{\alpha}}, x^{\hat{\alpha}}$	coordinates: general, (locally) inertial, SLI, SLI-gauged initially proper then comoving
x^α; ξ^μ , $\hat{\xi}^\mu$	(proper) position ; gauge transformation shift, SLI-gauge transformation shift (1.10)
$\boldsymbol{Y}, Y^\alpha, Y_\alpha$	general vector (field), (contra)vector (field), covector (field)
$< f >$	spatial integral (of an abstract function: f) over $\underline{V}$ (2.4a)
$\|x^n\|$; $\\|T^{\mu\nu}\\|$	spatial distance from the origin as a norm ; integral norms of a tensor over $\underline{V}$
$\{\dots_{\mu\nu}\}^{\mu\nu}$	reformatting a covariant second-rank tensor to a contravariant second-rank tensor

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