

On Goldbach Conjecture: An Algebraic Formulation using Odd Prime Representation

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Abstract

This paper explores an algebraic formulation for representing even integers as the sum of two primes, in the context of the Strong Goldbach Conjecture. The conjecture states that every even integer greater than 2 can be expressed as the sum of two prime numbers. We establish that for even integers $N > 4$, both primes in the sum must be odd. This allows us to utilize the property that all odd prime numbers can be represented in the form $2a + 1$ for some non-negative integer a . The fundamental challenge in proving the Goldbach Conjecture within this framework lies in demonstrating the primality of the resulting addends. The specific case of $N = 4$ is handled separately as $4 = 2 + 2$.

1 Introduction

The Goldbach Conjecture, specifically its strong form, is one of the oldest and best-known unsolved problems in number theory. It posits that every even integer greater than 2 is the sum of two prime numbers. For instance, $4 = 2 + 2$, $6 = 3 + 3$, $8 = 3 + 5$, and so on. Despite extensive computational verification for numbers up to 4×10^{18} [1], a formal mathematical proof remains elusive. This paper presents an algebraic framework for approaching the conjecture, highlighting the core difficulty that needs to be addressed for a complete proof.

2 Addressing the Prime Number 2

The prime number 2 is unique as it is the only even prime. When considering the sum of two primes, $N = P_1 + P_2$, for an even integer $N > 2$, there are two possibilities for the nature of the primes:

1. **Both P_1 and P_2 are odd primes.**
2. **One of the primes is 2.**

Let's examine the second case. If one of the primes, say P_1 , is 2, then $N = 2 + P_2$. Since N is an even number, P_2 must also be an even number ($N - 2 = P_2$). The only even prime number is 2 itself. Therefore, if one prime in the sum is 2, the other prime must also be 2. This implies that the only even integer $N > 2$ that can be expressed as a sum involving the prime 2 is $N = 2 + 2 = 4$.

For any even integer $N > 4$, if it were the sum of two primes and one of those primes was 2, then the other prime would have to be $N - 2$. However, $N - 2$ would be an even number greater than 2 (since $N > 4 \implies N - 2 > 2$), and no even number greater than 2 can be prime. Thus, for all even integers $N > 4$, if the Strong Goldbach Conjecture holds, N must be expressible as the sum of *two odd primes*.

3 Definitions and Formulation for $N > 4$

For even integers $N > 4$, we proceed with the representation of odd primes. Let P be an odd prime number. Any odd prime P can be uniquely expressed in the form:

$$P = 2a + 1$$

where a is a non-negative integer. Specifically, a can be derived from P as $a = \frac{P-1}{2}$.

Consider an even integer $N > 4$ that we wish to express as the sum of two odd primes, P_1 and P_2 . Let $P_1 = 2a_1 + 1$ and $P_2 = 2a_2 + 1$, where $a_1 = \frac{P_1-1}{2}$ and $a_2 = \frac{P_2-1}{2}$ are non-negative integers.

The sum N can then be written as:

$$N = P_1 + P_2$$

Substituting the $2a + 1$ form:

$$N = (2a_1 + 1) + (2a_2 + 1)$$

$$N = 2a_1 + 2a_2 + 2$$

Let k represent the number of primes in the sum. For the Strong Goldbach Conjecture involving two odd primes, $k = 2$. The general sum for k odd primes can be expressed as:

$$S_k = \sum_{i=1}^k P_i = \sum_{i=1}^k (2a_i + 1) = 2 \left(\sum_{i=1}^k a_i \right) + k$$

For the Strong Goldbach Conjecture, specifically for $N > 4$ where $k = 2$:

$$N = 2(a_1 + a_2) + 2$$

Since N is an even integer, we can write $N = 2X$ for some integer $X = N/2$. Substituting this into the equation:

$$2X = 2(a_1 + a_2) + 2$$

Dividing by 2, we obtain a relationship between X and the a_i values:

$$X = a_1 + a_2 + 1$$

which can be rearranged to:

$$a_1 + a_2 = X - 1$$

4 Illustrative Example (Strong Goldbach Conjecture for $N > 4$)

Let's consider the even integer $N = 18$. Here, N is an even integer greater than 4. We want to find P_1, P_2 (both odd primes) such that $18 = P_1 + P_2$.

Using our formulation: $X = N/2 = 18/2 = 9$. We need to find non-negative integers a_1 and a_2 such that:

$$a_1 + a_2 = X - 1$$

$$a_1 + a_2 = 9 - 1$$

$$a_1 + a_2 = 8$$

We can systematically list possible pairs (a_1, a_2) that sum to 8 (assuming $a_1 \leq a_2$ without loss of generality, and $a_i \geq 0$ as primes are positive):

a_1	a_2	$P_1 = 2a_1 + 1$	$P_2 = 2a_2 + 1$	Are P_1, P_2 Prime?	Sum $P_1 + P_2$
0	8	$2(0) + 1 = 1$	$2(8) + 1 = 17$	(1 is not prime)	18
1	7	$2(1) + 1 = 3$	$2(7) + 1 = 15$	(15 is not prime)	18
2	6	$2(2) + 1 = 5$	$2(6) + 1 = 13$	Yes (5, 13)	18
3	5	$2(3) + 1 = 7$	$2(5) + 1 = 11$	Yes (7, 11)	18
4	4	$2(4) + 1 = 9$	$2(4) + 1 = 9$	(9 is not prime)	18

This example successfully demonstrates that for $N = 18$, there exist pairs of non-negative integers (a_1, a_2) whose corresponding values $2a_1 + 1$ and $2a_2 + 1$ are prime numbers, thus satisfying the Goldbach Conjecture for this specific case.

5 The Core Problem and Future Direction

As highlighted by the illustrative example, the algebraic formulation successfully translates the sum of primes into a sum of derived integers a_i . The central challenge in proving the Strong Goldbach Conjecture using this approach is not in finding integer solutions for $a_1 + a_2 = X - 1$ (as there are generally many such solutions for a given X).

The **core problem** lies in demonstrating that, for *every* even integer $N > 4$, there will *always* exist at least one pair of non-negative integers (a_1, a_2)

satisfying $a_1 + a_2 = X - 1$ such that *both* $P_1 = 2a_1 + 1$ and $P_2 = 2a_2 + 1$ are **prime numbers**. This requires a deeper understanding of the distribution of prime numbers.

Future work would involve exploring more advanced number theoretic techniques, such as those from partition theory or sieve methods, which deal with the density and properties of primes, to either constrain the choices of a_1 and a_2 or provide a density argument to guarantee the existence of such prime pairs for all even N .

References

- [1] T. Oliveira e Silva, S. Herzog, S. Pardi, "Empirical verification of the Goldbach Conjecture and computation of Goldbach partitions up to 4×10^{18} ", *Mathematics of Computation*, vol. 83, no. 287, pp. 1521-1525, 2014.