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Conic and Cyclidic Sections in Double Conformal Geometric Algebra $\mathcal{G}_{8,2}$

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Abstract: The $\mathcal{G}_{8,2}$ Geometric Algebra, also called the Double Conformal / Darboux Cyclide Geometric Algebra (DCGA), has entities that represent conic sections. DCGA also has entities that represent planar sections of Darboux cyclides, which are called cyclidic sections in this paper. This paper presents these entities and many operations on them. Operations include reflection, projection, rejection, and intersection with respect to spheres and planes. Other operations include rotation, translation, and dilation. Possible applications are introduced that include orthographic and perspective projections of conic sections onto view planes, which may be of interest in computer graphics or other computational geometry subjects.

Keywords: conformal geometric algebra, conic sections, perspective projection

1. Introduction

The $\mathcal{G}_{8,2}$ Geometric Algebra, called the Double Conformal / Darboux Cyclide Geometric Algebra (DCGA), is introduced in [1][2]. This paper presents some additional results in DCGA for representing conic and cyclidic sections and for operating on conic and cyclidic sections in DCGA.

Conic sections and meet intersections were studied in conformal geometric algebra in [3]. The following sections of this paper elaborate on the conic and cyclidic sections with many illustrative figures produced using the Geometric Algebra Computing software called the Geometric Algebra ALgorithms Optimizer *Gaalop* that is introduced in [4]. In [2], it is explained how to use the Gaalop Visualizer to visualize DCGA entities. The Gaalop Visualizer is introduced in [5], it is based on CLUViz [6]. At the time of writing this paper, Gaalop appeared to be a very unique software for its ability to render visualizations of the DCGA conic and cyclidic section entities.

2. DCGA GIPNS section

Conformal geometric algebra in its modern form started with the work of D. Hestenes [7]. The concepts and definitions of geometric inner product null space (GIPNS) and geometric outer product null space (GOPNS) *entities* are explained in detail by Perwass in [8]. In this paper, all geometric entities are DCGA GIPNS entities.

The DCGA GIPNS 2-vector *2D-surface* entity $\boldsymbol{\Upsilon}$ is defined in [2] as generally being an instance of the DCGA GIPNS 2-vector *Darboux cyclide* surface entity $\boldsymbol{\Omega}$ or an instance of a *degenerate* form of $\boldsymbol{\Omega}$.
The first degenerate forms are the DCGA GIPNS 2-vector *Dupin cyclide* surface entity $\boldsymbol{\Phi}$ and the DCGA 2-vector *horned Dupin cyclide* surface entity $\boldsymbol{\Gamma}$. The next degenerate form is the DCGA GIPNS 2-vector *parabolic cyclide* surface entity $\boldsymbol{\Psi}$. Further degenerates of the parabolic cyclide entity $\boldsymbol{\Psi}$ are the DCGA 2-vector quadric surface entities. The Darboux cyclide $\boldsymbol{\Omega}$ and Dupin cyclide $\boldsymbol{\Phi}, \boldsymbol{\Gamma}$ entities are generally quartic surfaces. The parabolic cyclide $\boldsymbol{\Psi}$ entity is generally a cubic surface.

All of the DCGA GIPNS *2D-surface* entities $\boldsymbol{\Upsilon}$ can be intersected with a standard DCGA GIPNS 2-vector *plane* entity $\boldsymbol{\Pi}$ or with a standard DCGA GIPNS 2-vector *sphere* entity $\mathbf{S}$. The DCGA GIPNS 4-vector *intersection* 1D-surface entity $\boldsymbol{\Upsilon} \wedge \boldsymbol{\Pi}$ can also be called a DCGA GIPNS *section* entity

$$\psi = \boldsymbol{\Upsilon} \wedge \boldsymbol{\Pi} \tag{1}$$

A *1D-surface* on a plane is also called a *plane curve*. The degree of a plane curve or section ψ depends on the degree of the 2D-surface $\boldsymbol{\Upsilon}$ it is cut from. A section of a quartic surface $\boldsymbol{\Upsilon} = \boldsymbol{\Omega}, \boldsymbol{\Phi}$ or $\boldsymbol{\Gamma}$ is a quartic plane curve.
A section of a cubic surface, such as a parabolic cyclide $\boldsymbol{\Upsilon} = \boldsymbol{\Psi}$, is a cubic plane curve. A section of a quadric surface, such as a cone $\mathbf{K}$, is a quadratic plane curve, also called a *conic section*.

A DCGA GIPNS 4-vector *section* 1D-surface plane-curve entity ψ is defined as the intersection (1) of a DCGA GIPNS 2-vector *2D-surface* entity $\boldsymbol{\Upsilon}$ and a standard DCGA GIPNS 2-vector plane entity $\boldsymbol{\Pi}$.

As a special case, if $\boldsymbol{\Upsilon}$ is a standard DCGA GIPNS 2-vector *sphere* $\mathbf{S}$ or *plane* $\boldsymbol{\Pi}$, then $\psi = \boldsymbol{\Upsilon} \wedge \boldsymbol{\Pi}$ is either a standard DCGA GIPNS 4-vector *circle* entity $\mathbf{C}$ or a standard DCGA GIPNS 4-vector *line* entity $\mathbf{L}$. These entities are special since in our model they can be further intersected with each other, while other entities cannot be intersected with each other. These are

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special conic sections of a standard sphere or plane.

2.1 DCGA GIPNS conic section

Conic sections are planar cuts through a quadric cone surface. A *conic section* is an ellipse, parabola, hyperbola, line, non-parallel lines pair, or the cone vertex point. Planar cuts through other quadric surfaces, such as ellipsoid, hyperboloid, and paraboloid, can produce some of the conic section quadratic plane curves, but not all of them.

The DCGA GIPNS 4-vector *conic section* quadratic 1D-surface plane-curve entity $\boldsymbol{\kappa}$ is defined as the intersection

$$\boldsymbol{\kappa} \quad = \quad \mathbf{K} \wedge \mathbf{\Pi} \tag{2}$$

of a DCGA GIPNS 2-vector *cone* quadric 2D-surface entity $\mathbf{K}$ and a standard DCGA GIPNS 2-vector *plane* entity $\mathbf{\Pi}$. The cone entity $\mathbf{K}$ is a degenerate form of the parabolic cyclide entity $\mathbf{\Psi}$.

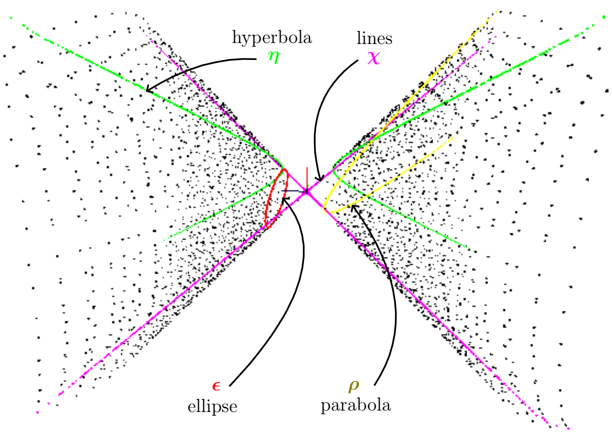

Figure 1. Conic sections: quadratic plane curves

Figure 1 shows some DCGA *conic section* entities $\boldsymbol{\epsilon}, \boldsymbol{\rho}, \boldsymbol{\eta}, \boldsymbol{\chi}$ rendered by the Gaalop Visualizer. The conic section entities are intersections $\mathbf{K} \wedge \mathbf{\Pi}$ of various standard DCGA *plane* entities $\mathbf{\Pi}$ with a DCGA *cone* entity $\mathbf{K}$.

In [2], the ellipse $\boldsymbol{\epsilon}^{||xy} = \mathbf{H}^{||z} \wedge \mathbf{\Pi}^{z=0}$, parabola $\boldsymbol{\rho}^{||xy} = \mathbf{B}^{||z} \wedge \mathbf{\Pi}^{z=0}$, and hyperbola $\boldsymbol{\eta}^{||xy} = \mathbf{J}^{||z} \wedge \mathbf{\Pi}^{z=0}$ in the xy-plane $(z = 0)$ are defined as the intersection of the xy-plane $\mathbf{\Pi}^{z=0}$ with a z-axis aligned elliptic cylinder $\mathbf{H}^{||z}$, parabolic cylinder $\mathbf{B}^{||z}$, and hyperbolic cylinder $\mathbf{J}^{||z}$, respectively.

2.2 DCGA GIPNS cyclidic section

Cyclidic sections are planar cuts through Darboux cyclide, Dupin cyclide, and parabolic cyclide surfaces. Since quadric surfaces are degenerate cyclides, the cyclidic sections are a generalization of the conic sections. Cyclidic sections through Darboux cyclides and Dupin cyclides are quartic plane curves. Cyclidic sections through parabolic cyclides are cubic plane curves. Conic sections through degenerate parabolic cyclides are quadratic or linear plane curves. The general cyclidic section can be represented by the intersection of a Darboux cyclide and plane.

The DCGA GIPNS 4-vector *cyclidic section* quartic 1D-surface plane-curve entity $\boldsymbol{\omega}$ is defined as the intersection

$$\boldsymbol{\omega} \quad = \quad \mathbf{\Omega} \wedge \mathbf{\Pi} \tag{3}$$

of a DCGA GIPNS 2-vector *Darboux cyclide* quartic 2D-surface entity $\mathbf{\Omega}$ and a standard DCGA GIPNS 2-vector *plane* entity $\mathbf{\Pi}$.

The entity $\mathbf{\Omega}$ may be a degenerate form that includes Dupin cyclides $\mathbf{\Phi}, \mathbf{\Gamma}$ and parabolic cyclides $\mathbf{\Psi}$, but not degenerate parabolic cyclides or quadric surfaces.
A *cyclidic section* entity $\boldsymbol{\omega}$ represents either a quartic or cubic plane curve. A quadratic or linear plane curve is a *conic section* entity $\boldsymbol{\kappa}$, which may also be called a *degenerate cyclidic section*.

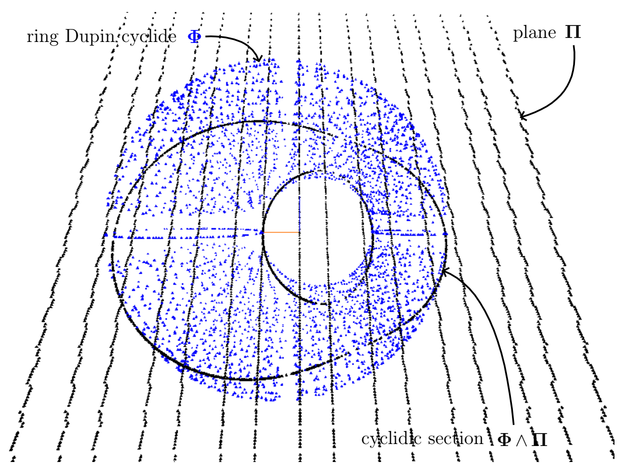

Figure 2. Cyclidic section: quartic plane curve.

Figure 2 shows an example of a cyclidic section that is a quartic plane curve. The DCGA *ring Dupin cyclide* $\mathbf{\Phi}$ $(R = 3, r_1 = 1, r_2 = 2)$ is cut by a DCGA plane $\mathbf{\Pi}_i$ $(\mathbf{n} = \mathbf{e}_3, d = 0)$ that is rotated $25°$ around the x-axis $\mathbf{e}_1$ using a DCGA rotor R operation as $\mathbf{\Pi} = R\mathbf{\Pi}_i R^{\sim}$. The cyclidic section entity is the intersection entity $\boldsymbol{\omega} = \mathbf{\Phi} \wedge \mathbf{\Pi}$.

Figure 3 shows an example of a cyclide section that is a cubic plane curve. The DCGA GIPNS 2-vector *parabolic cyclide* $\mathbf{\Psi} = \mathbf{SOS}^{\sim}$ is the DCGA GIPNS 2-vector *toroid* $\mathbf{O}$ $(R = 3, r = 2)$ reflected in the standard DCGA 2-vector *sphere* $\mathbf{S}$ $(\mathbf{p} = \mathbf{e}_1, r = 2)$. The sphere $\mathbf{S}$ center point $\mathbf{P}_{\mathcal{D}} = \mathcal{D}(\mathbf{p})$ is on the toroid $\mathbf{O}$ surface, and the parabolic cyclide $\mathbf{\Psi}$ is the *inversion* of the toroid $\mathbf{O}$ in the sphere $\mathbf{S}$. The

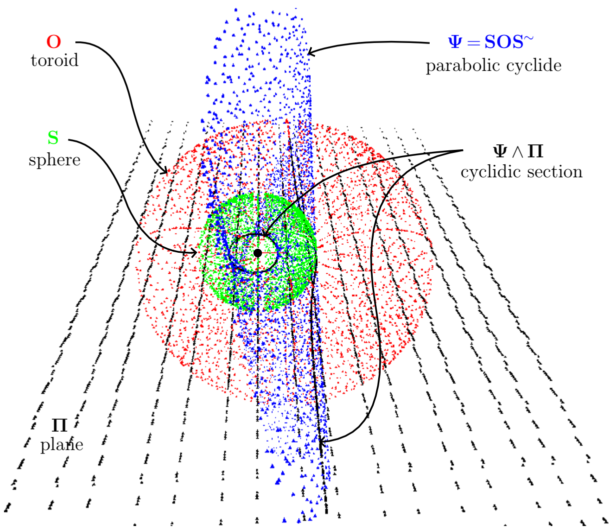

Figure 3. Cyclidic section: cubic plane curve.

cyclidic section $\mathbf{\Psi} \wedge \mathbf{\Pi}$ is the intersection of the parabolic cyclide $\mathbf{\Psi}$ and a standard DCGA *plane* $\mathbf{\Pi}$ ($\mathbf{n} = \mathbf{e}_3, d = 0$) that is rotated $25°$ around the x-axis $\mathbf{e}_1$.

3. Operations on DCGA GIPNS section entities

Operations on DCGA GIPNS section entities ψ include:

- *reflection* of any section entity ψ *in a sphere* $\mathbf{S}$ *or plane* $\mathbf{\Pi}$
- *projection* of any section entity ψ *onto a sphere* $\mathbf{S}$ *or plane* $\mathbf{\Pi}$
- *rejection* of any section entity ψ *from a sphere* $\mathbf{S}$ *or plane* $\mathbf{\Pi}$
- *intersection* of any section entity ψ *with a sphere* $\mathbf{S}$ *or plane* $\mathbf{\Pi}$
- and general *rotation*, *translation*, and *dilation*.

The reflection in a sphere is also known as inversion *in* a sphere. Inversion of a curve in a sphere produces the curve with points that are at an inverse displacement from the sphere center point when the sphere radius is $r = 1$, which is as expected. Reflection of a curve in a plane produces the reflected image of the curve on the far side of the reflection plane, which is also as expected.

The projection is a spherical or orthographic projection *onto* a sphere or plane surface, respectively. The inversion and projection of a curve point *in* and *onto* a sphere are collinear, on a line from the center of the sphere to the curve point. A curve point projects onto a sphere where the line through the curve point and its inverse point intersects the sphere surface. The projection line also passes through the sphere center point. A curve point projects onto a plane where the line through the curve point and its reflected point orthogonally intersects the plane surface, and the projected point is midway between the curve point and its reflected point.

The rejection is a perpendicular projection of a curve or surface from a sphere or plane, and it emerges at $90°$ or *normal to* the sphere or plane when the curve or surface and the sphere or plane have an intersection. The rejection produces a projected curve on the surface of a perpendicular plane or perpendicular sphere through intersection points. The rejection of a surface from a plane or sphere produces a rejected surface that is perpendicular. For example, an ellipsoid that intersects a plane can be rejected from the plane to produce an elliptic cylinder representing the ellipse cut through the ellipsoid by the plane.

The intersection of the section entities ψ with each other in general, not just with standard DCGA planes, spheres, lines, and circles, *would be* a very useful operation but unfortunately it does not work in our model. The DCGA intersections rule and formula is given in [2], which states that *only a single* entity that is not a standard DCGA plane $\mathbf{\Pi}$, sphere $\mathbf{S}$, line $\mathbf{L}$, or circle $\mathbf{C}$ can be included in a wedge that forms an intersection entity. This rule or limitation still applies when forming intersections that include a DCGA GIPNS section entity ψ. This means that it is possible to intersect any section entity $\psi = \mathbf{\Upsilon} \wedge \mathbf{\Pi}$ with other coplanar *circles* $\mathbf{C} = \mathbf{S} \wedge \mathbf{\Pi}$ and *lines* $\mathbf{L} = \mathbf{\Pi}_2 \wedge \mathbf{\Pi}$ as, for example, an intersection entity such as $\mathbf{\Upsilon} \wedge \mathbf{S} \wedge \mathbf{\Pi}_2 \wedge \mathbf{\Pi}$, which is an 8-vector. When the common plane $\mathbf{\Pi}$ is known, it can be contracted out of entities, and the example intersection could be written as $(\psi \cdot \mathbf{\Pi}) \wedge (\mathbf{C} \cdot \mathbf{\Pi}) \wedge (\mathbf{L} \cdot \mathbf{\Pi}) \wedge \mathbf{\Pi}$.

Although the operations are limited to working against only spheres and planes, they still allow for many interesting possibilities to produce transformed curves and surfaces that may have scientific, technical design and development or artistic uses. The following subsections explore these operations in more detail, with many illustrative figures rendered by the Gaalop Visualizer.

3.1 Reflection

The reflection ψ' of a DCGA GIPNS 4-vector *section* 1D-surface plane-curve entity ψ in a standard DCGA GIPNS 2-vector *plane* entity $\mathbf{\Pi}$ is defined as

$$\psi' = \mathbf{\Pi} \psi \mathbf{\Pi}^{\sim}. \tag{4}$$

The reflection, also called the inversion, ψ' of a DCGA GIPNS 4-vector *section* 1D-surface plane-curve entity ψ in a standard DCGA GIPNS 2-vector *sphere* entity $\mathbf{S}$ is defined as

$$\psi' = \mathbf{S} \psi \mathbf{S}^{\sim}. \tag{5}$$

Figure 4 shows the reflection (inversion) of a cyclidic section in a sphere.

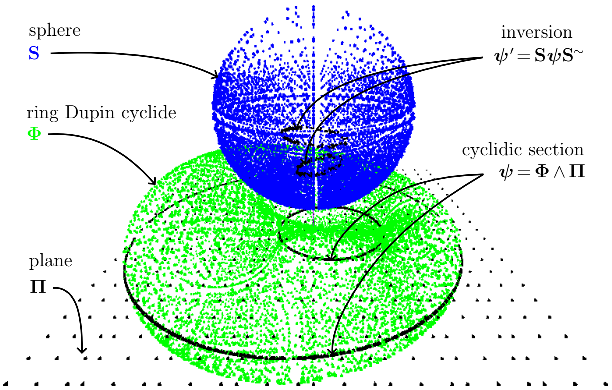

Figure 4. Reflection (inversion) of cyclidic section in sphere

3.2 Projection

The projection ψ' of a DCGA GIPNS 4-vector *section* 1D-surface plane-curve entity ψ onto a standard DCGA GIPNS 2-vector *plane* entity $\mathbf{\Pi}$ is defined as

$$\psi' = (\psi \cdot \mathbf{\Pi})\mathbf{\Pi}^{-1} = (\psi \cdot \mathbf{\Pi})\mathbf{\Pi}^{\sim}. \tag{6}$$

The projection ψ' of a DCGA GIPNS 4-vector *section* 1D-surface plane-curve entity ψ onto a standard DCGA GIPNS 2-vector *sphere* entity $\mathbf{S}$ with radius r is defined as

$$\psi' = (\psi \cdot \mathbf{S})\mathbf{S}^{-1} = \frac{1}{r^4}(\psi \cdot \mathbf{S})\mathbf{S}^{\sim}. \tag{7}$$

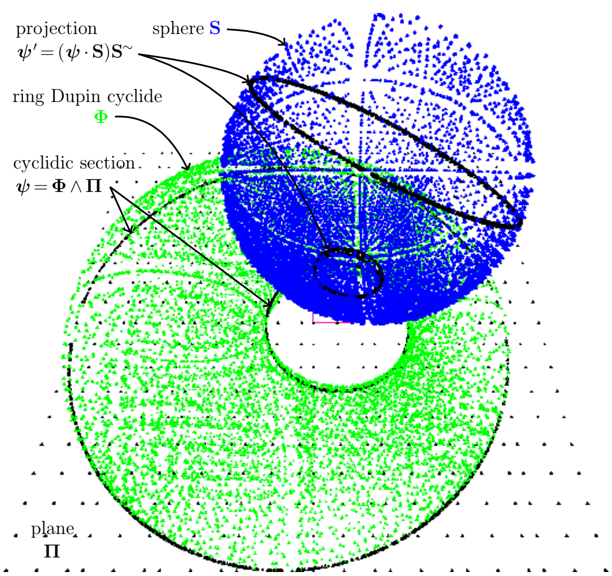

Figure 5. Projection of cyclidic section onto sphere

Figure 5 shows the projection of a cyclidic section onto a sphere.

3.3 Rejection

The rejection ψ' of a DCGA GIPNS 4-vector *section* 1D-surface plane-curve entity ψ from a standard DCGA GIPNS 2-vector *plane* entity $\mathbf{\Pi}$ is defined as

$$\psi' = (\psi \wedge \mathbf{\Pi})\mathbf{\Pi}^{-1} = (\psi \wedge \mathbf{\Pi})\mathbf{\Pi}^{\sim}. \tag{8}$$

The rejection ψ' of a DCGA GIPNS 4-vector *section* 1D-surface plane-curve entity ψ from a standard DCGA GIPNS 2-vector *sphere* entity $\mathbf{S}$ with radius r is defined as

$$\psi' = (\psi \wedge \mathbf{S})\mathbf{S}^{-1} = \frac{1}{r^4}(\psi \wedge \mathbf{S})\mathbf{S}^{\sim}. \tag{9}$$

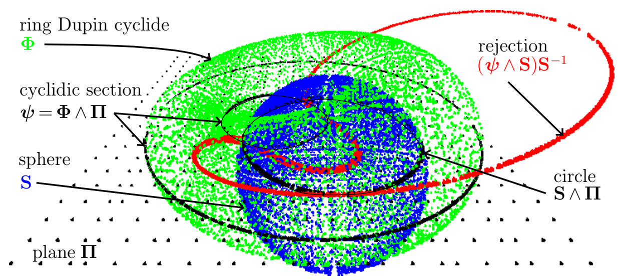

Figure 6. Rejection of cyclidic section from sphere

Figure 6 shows the rejection of a cyclidic section from a sphere.

It may be difficult to see from the figure, but the rejection curve (red) intersects through the sphere at a 90° angle to the surface.

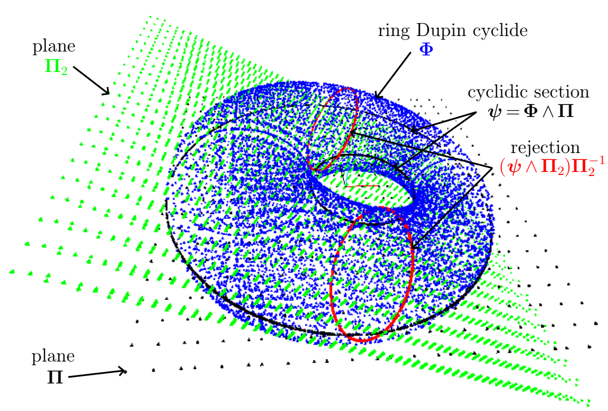

Figure 7. Rejection of cyclidic section from plane

Figure 7 shows the rejection of a cyclidic section from a plane.

3.4 Commutator and anti-commutator projections

The geometric product AB can be written as the sum of the anti-symmetric *commutator* product $\times$ and the symmetric *anti-commutator* product $\bar{\times}$ as

$$\begin{aligned} AB &= \frac{1}{2}(AB - BA) + \frac{1}{2}(AB + BA) \quad (10)\\ &= A \times B + A \bar{\times} B. \quad (11) \end{aligned}$$

Given a cyclidic section $\boldsymbol{\psi}$ and a sphere $\mathbf{S}$ (or a plane), then

$$\begin{aligned} \boldsymbol{\psi} &= \boldsymbol{\psi}\mathbf{S}\mathbf{S}^{-1} \quad (12)\\ &= \left(\frac{1}{2}(\boldsymbol{\psi}\mathbf{S} - \mathbf{S}\boldsymbol{\psi}) + \frac{1}{2}(\boldsymbol{\psi}\mathbf{S} + \mathbf{S}\boldsymbol{\psi})\right)\mathbf{S}^{-1} \quad (13)\\ &= (\boldsymbol{\psi} \times \mathbf{S})\mathbf{S}^{-1} + (\boldsymbol{\psi} \bar{\times} \mathbf{S})\mathbf{S}^{-1}. \quad (14) \end{aligned}$$

The *commutator projection* of $\boldsymbol{\psi}$ onto $\mathbf{S}$ can be defined as

$$\boldsymbol{\psi}' = (\boldsymbol{\psi} \times \mathbf{S})\mathbf{S}^{-1} \quad (15)$$

and the *anti-commutator rejection* of $\boldsymbol{\psi}$ from $\mathbf{S}$ can be defined as

$$\boldsymbol{\psi}' = (\boldsymbol{\psi} \bar{\times} \mathbf{S})\mathbf{S}^{-1}. \quad (16)$$

As an example, the next figure looks at a parabola and its reflection, projection, rejection, commutator projection, and anti-commutator rejection with respect to a sphere.

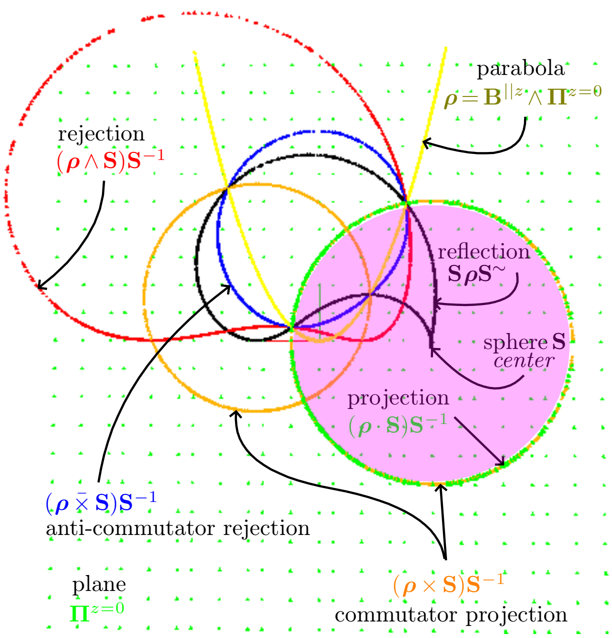

Figure 8. Spherical operations on a parabola

Figure 8 shows spherical operations on a parabola. The sphere $\mathbf{S}$, not rendered, is enclosed in the green circle and is shaded magenta. The parabola $\boldsymbol{\rho}$ is constructed as the intersection of a z-axis aligned parabolic cylinder $\mathbf{B}^{||z}$ and the xy-plane $\boldsymbol{\Pi}^{z=0}$.

If the plane $\boldsymbol{\Pi}^{z=0}$ is contracted out of any of the projection or rejection plane curves, then the result is a surface that contains the plane curve. For example, the *anti-commutator rejection* plane curve can be turned into a DCGA GIPNS *2D-surface* $\boldsymbol{\Upsilon}$ as

$$\begin{aligned} \boldsymbol{\Upsilon} &= \boldsymbol{\Pi}^{z=0} \cdot ((\boldsymbol{\rho} \bar{\times} \mathbf{S})\mathbf{S}^{-1}) \quad (17)\\ &= \boldsymbol{\Pi}^{z=0} \rfloor ((\boldsymbol{\rho} \bar{\times} \mathbf{S})\mathbf{S}^{-1}). \quad (18) \end{aligned}$$

3.5 Intersection

As explained in [2], all DCGA entities can be intersected only with standard DCGA spheres $\mathbf{S}$ and planes $\boldsymbol{\Pi}$. The standard DCGA line $\mathbf{L} = \boldsymbol{\Pi}_2 \wedge \boldsymbol{\Pi}$ and circle $\mathbf{C} = \mathbf{S} \wedge \boldsymbol{\Pi}$ entities are constructed from standard sphere $\mathbf{S}$ and plane $\boldsymbol{\Pi}$ entities. The set $\mathcal{S} = \{\mathbf{S}, \boldsymbol{\Pi}\}$ includes all instances of the *standard bi-CGA GIPNS 2-vector entities*, which are spheres and planes. All of the plane-curve *section* entities $\boldsymbol{\psi} = \boldsymbol{\Upsilon} \wedge \boldsymbol{\Pi}$ can be intersected with coplanar lines $\mathbf{L} = \boldsymbol{\Pi}_2 \wedge \boldsymbol{\Pi}$ and circles $\mathbf{C} = \mathbf{S} \wedge \boldsymbol{\Pi}$, but in our model not with any other types of coplanar curves. Coplanar curves may intersect in four or less points in the plane $\boldsymbol{\Pi}$.

The DCGA GIPNS 6-vector *intersection* $\mathbf{X}$ of a DCGA GIPNS 4-vector *section* 1D-surface plane-curve entity $\boldsymbol{\psi} = \boldsymbol{\Upsilon} \wedge \boldsymbol{\Pi}$ and a coplanar standard DCGA GIPNS 4-vector *line* $\mathbf{L} = \boldsymbol{\Pi}_2 \wedge \boldsymbol{\Pi}$ in the plane $\boldsymbol{\Pi}$ is defined as

$$\begin{aligned} \mathbf{X} &= (\boldsymbol{\psi} \cdot \boldsymbol{\Pi}) \wedge (\mathbf{L} \cdot \boldsymbol{\Pi}) \wedge \boldsymbol{\Pi} \quad (19)\\ &\simeq \boldsymbol{\Upsilon} \wedge \boldsymbol{\Pi}_2 \wedge \boldsymbol{\Pi}. \quad (20) \end{aligned}$$

The DCGA GIPNS 6-vector *intersection* $\mathbf{X}$ of a DCGA GIPNS 4-vector *section* 1D-surface plane-curve entity $\boldsymbol{\psi} = \boldsymbol{\Upsilon} \wedge \boldsymbol{\Pi}$ and a coplanar standard DCGA GIPNS 4-vector *circle* $\mathbf{C} = \mathbf{S} \wedge \boldsymbol{\Pi}$ in the plane $\boldsymbol{\Pi}$ is defined as

$$\begin{aligned} \mathbf{X} &= (\boldsymbol{\psi} \cdot \boldsymbol{\Pi}) \wedge (\mathbf{C} \cdot \boldsymbol{\Pi}) \wedge \boldsymbol{\Pi} \quad (21)\\ &\simeq \boldsymbol{\Upsilon} \wedge \mathbf{S} \wedge \boldsymbol{\Pi}. \quad (22) \end{aligned}$$

If $\boldsymbol{\gamma}_1 = (\mathbf{B}_1 \in \mathcal{S}) \wedge \boldsymbol{\Pi}$ and $\boldsymbol{\gamma}_2 = (\mathbf{B}_2 \in \mathcal{S}) \wedge \boldsymbol{\Pi}$ are standard DCGA GIPNS 4-vector *circle* $\mathbf{C}$ or *line* $\mathbf{L}$ entities in the plane $\boldsymbol{\Pi}$, then their DCGA GIPNS 8-vector *intersection* $\mathbf{X}$ with a coplanar DCGA GIPNS 4-vector *section* 1D-surface plane-curve entity $\boldsymbol{\psi} = \boldsymbol{\Upsilon} \wedge \boldsymbol{\Pi}$ can be defined as

$$\begin{aligned} \mathbf{X} &= (\boldsymbol{\psi} \cdot \boldsymbol{\Pi}) \wedge (\boldsymbol{\gamma}_1 \cdot \boldsymbol{\Pi}) \wedge (\boldsymbol{\gamma}_2 \cdot \boldsymbol{\Pi}) \wedge \boldsymbol{\Pi} \quad (23)\\ &\simeq \boldsymbol{\Upsilon} \wedge \mathbf{B}_1 \wedge \mathbf{B}_2 \wedge \boldsymbol{\Pi}. \quad (24) \end{aligned}$$

4. Applications

4.1 Orthographic projection of a conic section

The orthographic projection $\boldsymbol{\kappa}_{\text{ortho}}$ of a DCGA GIPNS 4-vector *conic section* entity $\boldsymbol{\kappa} = \mathbf{K} \wedge \boldsymbol{\Pi}_{\boldsymbol{\kappa}}$ onto a standard DCGA GIPNS 2-vector *plane* $\boldsymbol{\Pi}$ is defined as

$$\boldsymbol{\kappa}_{\text{ortho}} = (\boldsymbol{\kappa} \cdot \boldsymbol{\Pi})\boldsymbol{\Pi}^{-1}. \quad (25)$$

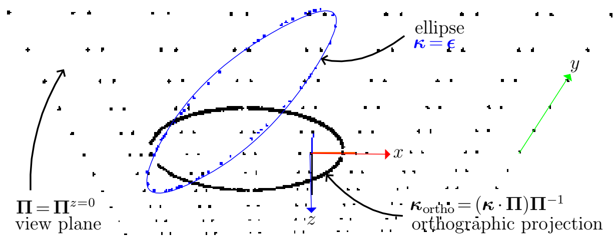

Figure 9. Orthographic projection of ellipse

Figure 9 shows an orthographic projection $\boldsymbol{\kappa}_{\text{ortho}}$ of an ellipse $\boldsymbol{\kappa} = \boldsymbol{\epsilon}$ onto a view plane $\boldsymbol{\Pi} = \boldsymbol{\Pi}^{z=0}$. The ellipse $\boldsymbol{\epsilon}$, which is constructed as the intersection of an ellipsoid and a plane, is rotated by 45° around the y-axis.

4.2 Perspective projection of a conic section

The perspective projection $\boldsymbol{\kappa}_{\text{persp}}$ of a DCGA GIPNS 4-vector *conic section* entity $\boldsymbol{\kappa} = \mathbf{K} \wedge \boldsymbol{\Pi}_{\boldsymbol{\kappa}}$ onto a standard DCGA GIPNS 2-vector *plane* $\boldsymbol{\Pi}$ from the view point $\mathbf{p} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ represented by a standard DCGA GIPNS 2-vector *sphere* $\mathbf{S}$ with center $\mathbf{P}_{\mathcal{D}} = \mathcal{D}(\mathbf{p})$ can be defined as

$$\begin{aligned} \boldsymbol{\kappa}_{\text{persp}} &= (((\boldsymbol{\kappa}\cdot\mathbf{S})\mathbf{S}^{-1})\cdot\mathbf{S})\wedge\boldsymbol{\Pi} && (26)\\ &= (\mathbf{S}\rfloor((\boldsymbol{\kappa}\cdot\mathbf{S})\mathbf{S}^{-1}))\wedge\boldsymbol{\Pi} && (27)\\ &= \mathbf{K}_{\mathbf{p}}\wedge\boldsymbol{\Pi}. && (28) \end{aligned}$$

The conic section $\boldsymbol{\kappa}$ is projected onto the sphere $\mathbf{S}$ as $\boldsymbol{\kappa}' = (\boldsymbol{\kappa}\cdot\mathbf{S})\mathbf{S}^{-1}$. The sphere $\mathbf{S}$ is contracted out of $\boldsymbol{\kappa}'$ to form a cone $\mathbf{K}_{\mathbf{p}} = \boldsymbol{\kappa}'\cdot\mathbf{S} = \mathbf{S}\rfloor\boldsymbol{\kappa}'$. The cone $\mathbf{K}_{\mathbf{p}}$ has vertex point $\mathbf{P}_{\mathcal{D}} = \mathcal{D}(\mathbf{p})$ and it contains both curves $\boldsymbol{\kappa}$ and $\boldsymbol{\kappa}'$. The cone $\mathbf{K}_{\mathbf{p}}$ is intersected with any view plane $\boldsymbol{\Pi}$ to form the perspective projection. The radius $r \neq 0$ of sphere $\mathbf{S}$ is arbitrary, but $r = 1$ could be assumed.

It is also possible to use the reflection $\mathbf{S}\boldsymbol{\kappa}\mathbf{S}^{-1}$ and define $\boldsymbol{\kappa}_{\text{persp}}$ as

$$\begin{aligned} \boldsymbol{\kappa}_{\text{persp}} &= ((\mathbf{S}\boldsymbol{\kappa}\mathbf{S}^{-1})\cdot\mathbf{S})\wedge\boldsymbol{\Pi} && (29)\\ &= (\mathbf{S}\rfloor(\mathbf{S}\boldsymbol{\kappa}\mathbf{S}^{-1}))\wedge\boldsymbol{\Pi} && (30)\\ &= \mathbf{K}_{\mathbf{p}}\wedge\boldsymbol{\Pi}. && (31) \end{aligned}$$

Figure 10 shows a perspective projection $\boldsymbol{\kappa}_{\text{persp}}$ of a parabola $\boldsymbol{\kappa} = \boldsymbol{\rho}$ onto a view plane $\boldsymbol{\Pi}^{z=-1}$ from a view point (or eye point) $\mathbf{p} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ at the center $\mathbf{P}_{\mathcal{D}} = \mathcal{D}(\mathbf{p})$ of a sphere $\mathbf{S}$. The sphere $\mathbf{S}$, not rendered, is shaded. The parabola $\boldsymbol{\rho}$ is constructed as the intersection of a z-axis aligned parabolic cylinder $\mathbf{B}^{||z}$ and the xy-plane $\boldsymbol{\Pi}^{z=-2}$.

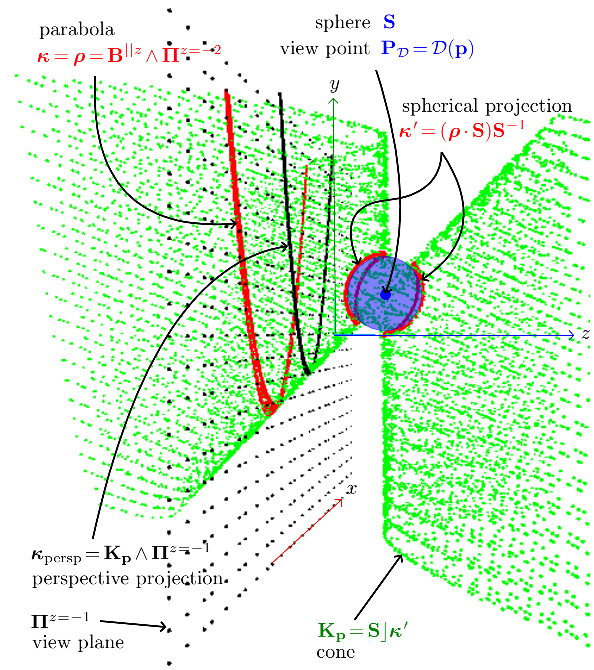

Figure 10. Perspective projection of parabola

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