School algebra as a surrounding container for school arithmetic

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Abstract

Algebra and arithmetic are contrasted from a perspective of school mathematics. The surrounding container view regarding algebra and arithmetic is formulated and defended. It is argued that an encounter with algebra may precede the introduction to arithmetic in school. In particular static process algebra is suggested as a theme which may play a useful role early on in the educational process heading towards the development of skills and competences in arithmetic.

*Minstroom Research BV, Utrecht, The Netherlands (hereafter called Minstroom Research), KvK nr. 59560347. Author's email address: `info@minstroomreserch.org`, `janaldertb@gmail.com`. Appendix A provides detailed statements concerning copyright protection of this document and about its status. This paper is a nopreprint in conformance with the definition given in Appendix B. Minstroom Research nopreprint series nr. 8 (Minstroom Research NPP#8, subseries number I4PSM#2). The paper has Minstroom Research document class B (see Appendix C).

Contents

1 Introduction

My interest in relating informatics with primary school mathematics (PSM) arose from investigating on how to assign a meaning to expressions which seem to call for division by zero such as $0/0, 1/0$, and 0^{-1}. For this work I refer to [9], [6] which was inspired by the discussion on school mathematics in [14], and to [8]. The presence and relevance of issues concerning division by zero in connection with the teaching of arithmetic was reconfirmed in [15], on pages 231-232.

Contemplating division by zero led to further reflection on the concept of fractions in [3] and [1]. The work on fractions in turn has led to the hypothesis that a paraconsistent reasoning strategy, such as for instance the so-called "chunk and permeate" strategy that was proposed in [10], is implicit in all school mathematics.

Alternatively but not without conceptual difficulties one may think in terms of feature interaction (see e.g. [12]) with values and forms playing the role of different features of arithmetical expressions. The mentioned hypothesis on the omnipresence of a paraconsistent background plays no role in the presentation below. It follows from this hypothesis, however, that a plausible encoding of reasoning patterns and strategies for school mathematics in a formal logic is hard to find. So-called classical two-valued logic seems not to work properly,[1] and simple remedies such as the use of a three-valued logic or the use sequential logical connectives provide only partial solutions. Determination of an informal logic that may be thought to constitute a basis for PSM appears to be unexpectedly difficult.

The work in this paper is mainly based on [2], which made use of [4]. From [2] the following assumptions and viewpoints may be extracted:

- I4PSM (Informatics for PSM) is worth being investigated as an option for further development of PSM.
- While PSM is trivial from the perspective of mathematics, it is far from trivial from the perspective of theoretical informatics.
- The aspects of PSM that are non-trivial from an informatics perspective may be of importance for PSM teaching, and more importantly for the development of teachable content matter close to PSM.
- A remarkable diversity of connections between initial aspect of PSM and informatics may be distinguished. The main aspects highlighted in [2] are:
    1. concurrent and nested containers; (static) process algebra,

[1] When working in the partial meadow of rational numbers (that is rational numbers with a name for the multiplicative inverse, which is interpreted as a partial function), one may hold that $\Phi \equiv \forall x(x \neq 0 \rightarrow x \cdot x^{-1} = 1$ is a valid assertion). The validity of Φ, however, entails, end even depends on, the validity of its substitution instance for $= 0$. In other words $0 \neq 0 \rightarrow 0 \cdot 0^{-1} = 1$ is true, and assuming that one is using a a two-valued logic it should be assumed that $0 \cdot 0^{-1} = 1$ is either true or false, both alternatives being unattractive assuming that 0^{-1} is considered to be undefined.

2. typing, polymorphism, type coercion, and paraconsistency,
3. operator families with multiple arities for repeated aplication of the same infix operator, (for avoiding the use of a commutative and associative binary operator),
4. finite arithmetics, and
5. closed world assumptions.

- I4PSM allows significantly more room than conventional PSM for free contemplation of content matter and for the creative use of formulae, expressions, equations, laws, and systems modelling by formal means.
- I4PSM is a counterpart to the Dutch phrase "rekenen-informatica".

In this paper I will propose a viewpoint on the relation between algebra and arithmetic and I will draw the following conclusions which I consider to be somewhat surprising:

> Static process algebra, that is process algebra in which components don't allow state transitions, may precede the introduction to arithmetic in I4PSM. Static process algebra introduces a semigroup structure based on parallel composition which has no natural embedding in a group structure.

2 Algebra versus arithmetic

A major difficulty that I experience when reading about PSM is the distinction between arithmetic and algebra. One gets the impression that many authors claim to know what this distinction amounts to and write accordingly, while an awareness that finding a demarcation between algebra and arithmetic might be problematic is largely absent.

Providing a comprehensive survey of viewpoints about this issue is not even easy and for my objectives doing so it is not essential. I consider the distinction between algebra and arithmetic as not having been conclusively settled in the educational literature on mathematics, and as a consequence I will allow myself to suggest a description of the relation between algebra and arithmetic which I intend to use in subsequent work on I4PSM.

I don't claim that this viewpoint is novel as I am insufficiently knowledgeable about the relevant literature to assess the originality of my viewpoint. But this weakness does not really matter for the objectives that I have in mind. I will make use of it and from the moment that I find out who to refer to for a more original source than this paper I will do so. I don't expect or intend to convince PSM authors and researchers, who hold different views on "algebra versus arithmetic" of the validity of my view on algebra versus arithmetic either as different views on the matter may coexist. Different views on the demarcation of algebra and arithmetic may even serve as a foundation for one and the same PSM curriculum assuming that making that very distinction constitutes no part of the PSM content proper.

2.1 Some preliminary considerations

There may be different views on algebra versus arithmetic, such may even compete. I will first make some remarks on the context of the demarcation between algebra and arithmetic.

1. Algebra has three meanings at least, as an mathematical object, as a reasoning method specific for mathematics, and as topic within mathematics:

   (a) An algebra is a mathematical structure with certain features. There are different views on what precisely these features are, one may prefer similarity types to the use of signatures or conversely, one may or may not consider many-sortedness, one may or may not accept empty sorts, one may view constants as functions, one may or may not accept relations, one may or may not insist to work up to isomorphism only. Some may insist that algebras not commonly studied in mathematics are to be referred to as universal algebras (and so is the field of studying those).

   (b) Algebra is a way of working with fragments of mathematical text ("after applying some algebra we obtain the following expressions"), and,

   (c) Algebra is a part of mathematics, and one may disagree on which part is meant. Some may consider universal algebra a chapter in algebra, some may consider algebra to constitute a chapter in universal algebra, some may consider both to be disjoint.

   For some authors algebra viewed as a topic starts as soon as algebraic structures outside the arithmetical structures are used, investigated, or defined. Here permutation groups, constitute motivating examples. For others algebra starts only on the basis of category theory with co-algebras rather than algebras as the principal concept. I assume that once non-injective morphisms between algebraic structures come into play one may certainly speak of algebra or of a topic involvings algebra.

2. I assume that in the context of PSM and at this stage of development both the existence of non-injective morphisms and the existence of categories and co-algebras need not be taken into account (though it might be in the future).

3. When demarcating algebra and arithmetic it is practical to formulate a view on analysis and topology as well. Analysis begins with the reals or formalisation thereof, while topology has an axiomatic basis abstracted from the topology one observes in the case of the reals. Linear algebra uses analysis (reals) to define algebras, in particular groups.

4. A chapter (piece, subtopic, theme, theory) of mathematics may combine algebra, arithmetic, analysis, and topology. If one of these aspects is dominant in the chapter, the entire chapter may be subsumed under algebra.

5. It is uncommon to view a structure that occurs in arithmetic as "an arithmetic". But one may consider arithmetical structures, and one may think of these as arithmetical algebras. By default, however, arithmetic is mainly used as a reference to certain methods and topics. Stated differently: arithmetic shares with algebra two of the three forms of meaning mentioned above.

2.2 Viewpoints that I prefer to reject

Viewpoints on algebra versus arithmetic that I could find usually explain under what circumstances a topic migrates from being classified as arithmetic to being subsumed under algebra. Or in other words, which boundaries mark the transition from arithmetic to algebra. Each of the following views can be found in the literature and on many websites, sometimes in combination. These views each formulate restrictions on mathematical activity (assertional or transformational) which when not observed signify a transition into algebra (viewed in accordance with that view).

I did not find any author who had doubts on his or her views on where the boundary between arithmetic and algebra is to be placed to the extent that it was deemed useful to introduce a name for the author's viewpoint in order for it to be easily distinguished from other views on the same matter. I will improve in that in three ways: (i) by giving a name to my preferred viewpoint, (ii) by listing and naming competing views, and (ii) by criticising my own view. Here is a listing of "other views":

Calculation view: The calculation view holds that arithmetic is (exclusively) about the calculation of values with a limited repertoire of operations including constants. In this view asserting $2+3 = 5$ qualifies as arithmetic whereas asserting $5 = 2 + 3$ is a matter of algebra.

An extreme version of the calculation view considers all calculation to be part of arithmetic. This "calculation is arithmetic view" is debatable: popping a stack may be viewed as calculation in a datatype of stacks, but it hardly counts as a arithmetic.

Specific values only view: only properties of closed expressions in a predefined syntax for numbers and fractions qualify as assertions in arithmetic.

This view is somewhat more permissive than the calculation view as $5 = 2 + 3$ will be considered an arithmetical assertion.

No-variables view: the no-variables view holds that the step from arithmetic to algebra is made when variables are used.

In this view one may say: "for all pairs of numbers, say 3 and 7 we find $3 \cdot 7 = 7 \cdot 3$. Indeed order does not matter in a multiplication." The latter assertion would would not classify a a part of arithmetic in the specific values only view.

No calculation with variables view: the no calculation with variables view holds that rewriting say $3 \cdot x = 2 \cdot x$ into x marks the transition from arithmetic into algebra. Thus (in this view) solving $2 + 3 \cdot x = 5 \cdot x$ requires algebra, while solving $2 + 3 \cdot x = 5$ does not.

No-laws view: the no-laws view holds that the transition from arithmetic to algebra is made when laws like commutativity and associativity come into play.

Someone subscribing to the no-laws view may consider the question "find a value for x such that $3 + x = 5$" a matter of arithmetic while someone subscribing to the no-variables view may classify that question (or what's needed for answering it) outside arithmetic.

Ordinal and cardinal number realism: (ordinal and cardinal) number realism assers that the natural numbers each exist, can be used as ordinals and as cardinals, and can be transformed with several key operations. After augmenting this arena of numbers with negative values and fractions one finds (all of) arithmetic.

This form of realism is philosophically outdated, structures rather than numbers exist in recent forms of mathematical realism.

According to the each of these views Peano's arithmetic is not about, or a part of, arithmetic. This simple argument refutes each of these views (in my view). Each of these views is too restrictive regarding arithmetic. These views need not be comprehensive for someone's particular view on the demarcation between algebra and arithmetic. A supporter of the no-variables view may or may not count calculation in the prime field modulo p an instance of arithmetic.

3 Surrounding container view on algebra versus arithmetic

I will baptise my personal view on algebra versus arithmetic as the surrounding container view. The surrounding container view holds that algebra is a container of arithmetic, or stated the other way around arithmetic is a specialisation of algebra, and that algebra surrounds arithmetic in the following sense: by moving away from arithmetic, in a variety of directions, one enters the territory of non-arithmetic algebra.

3.1 The surrounding container view in more detail

Here is a detailed exposition of the surrounding container view of algebra in its relation to arithmetic:

1. Algebra is a large container which includes arithmetic; it extends in many directions beyond arithmetic. Some parts of algebra have specific features which invite one to use a specific name: arithmetic, number theory, linear

algebra, group theory, universal algebra, ring and fields, semigroup theory, abstract datatypes.

The relation between physics and science is comparable: science is a surrounding container of physics.

2. Arithmetic is algebra because it consists of knowledge about a specific family of algebraic structures. This family is fuzzy in the sense that different persons may include different structures in it. The structures may be called arithmetical algebras. The mentioned family contains:

   - Natural numbers with a subset of the following constants and operations: zero, unit, successor, addition, multiplication, cut-off subtraction, integer division, ordering (viewed as a function), primitives for binary notation, primitives for decimal notation. For each signature one has another algebra (or isomorphism class of algebras), each counting as an arithmetical algebra.
   - Integers, with negation and subtraction as operators as well as the operations mentioned for naturals. Again for many sub signatures one finds tailored arithmetical algebras.
   - Rational numbers, with or without division. In case of division one deals with meadows of rational numbers for which several forms exist: involutive meadows, partial meadows, non-involutive meadows, common meadows, wheels, transrationals.
   - Galois fields, Rational complex numbers, ring of Gauss, quaternions. For each of these structures (with various sets of operators defined on them) different individual may disagree on their being arithmetical. I am inclined to view the ring of Gauss and the rational complex numbers as part of arithmetic an nothing else.

   I consider natural numbers, integers, and rational numbers (common meadows version) with various signatures as core arithmetical structures. The complex rationals for instance are not core arithmetic.

3. Non-arithmetical algebra (often simply referred to as algebra) is encountered at least in the following cases:

   - If groups are considered which are not (isomorphic to) subgroups of arithmetical algebras.
   - If classes of algebraic structures are considered that are models of sets of equations or conditional equations.
   - If categories of algebraic structures are considered.
   - If non-injective morphisms between algebraic structures are considered.
   - If term models are constructed from sets of equations or conditional equations (however this construction may be a case of algebra leading to arithmetic).

Each of the core arithmetical structures exists in (is surrounded by) a vast volume of "not quite arithmetical structures" which again is included in a large collection of clearly non-arithmetical structures. For instance by encoding Boolean values in a two (or more) element sort and having an equality function into that sort one constructs on the basis of an arbitrary arithmetical algebra another algebra which is not arithmetical by definition but which is at the same time almost arithmetical. Sharp boundaries seem to be unconvincing, whereas fuzzy boundaries don't quite support arithmetical talk. Perhaps one must be open to the idea that an algebra is, say 60% arithmetic.

3.2 Criticizing the surrounding container view

A weakness of the surrounding container view is that might be considered purely ad hoc. Why not include quaternions and so on.

Another weakness might be that does not reflect the structure of three meanings as mentioned above in 2.1, while arithmetic might be understood as featuring a similar structure of layered but disparate meanings. Perhaps each of the three mention forms of meaning of algebra gives rise to its own boundary with arithmetic. In that case the surrounding container view may provide no more than an ad hoc way to find a boundary in the context of "algebra as a topic", and it may not potentially compete with the so-called competing views listed above which in some cases must be understood in the context of "algebra as a method".

3.3 PSM trajectories through arithmetic

According to the surrounding container view all of conventional PSM is arithmetical. School mathematics offers a trajectory though some arithmetic which after some time (though not in the primary phase, however) includes algebra proper (that is algebra which is not viewed as arithmetic).

It is consistent with the surrounding container view of algebra and arithmetic, however, that PSM may start outside arithmetic with some algebra only to become a form of arithmetic in a subsequent stage. This argument may be decomposed in a chain of steps as follows:

1. There is no such thing as a natural number; what exists in a plausible form of mathematical realism, called structural realism, is one or more classes of structures (algebras) for natural numbers and expressions for denoting elements of such structures. 17 for instance is the function that chooses the 17th element from each natural number structure in one's favourite (or rather current) class of structures for naturals, or stated differently it is the mapping that assigns a meaning to an expression for 17 to each natural number structure.

2. The notion of an expression becomes central. We have numbers in order to find meanings for expressions instead of the other way around. There

is no concept of number which a student is supposed to understand. However, there is a concept of structure (arithmetical algebra) which must be taught.

3. Once expressions and formulae have been spotted as essential carriers of a mathematical story which is not about numbers (because these don't exist outside algebras) the question may be posed if a student's first confronation with syntax admitting interpretation in an algebra must be with syntax that is preferably interpreted in an arithmetical algebra.

4. Apart from the current massive pressure from PSM educational practice there is no obvious ground why within PSM a student's first confrontation with algebra must be an encounter with arithmetic (that is some particulararithmetical algebra).

5. I will indicate below how an entirely different form of algebra might be placed in front of known PSM (all arithmetical in nature) in order to develop a context in which the power of arithmetic may make more (or additional) sense to a student.

4 Concurrent containers and nested containers

With DPII (Deviant Preview on "Instap" and "Instroom") I will refer to [13]. Besides [11] which contains a convincing motive to consider finite approximations to arithmetical algebras, DPII was the primary source of inspiration for starting work on I4PSM in [2] at the initial level of mathematical teaching.

DPII explains the use of numbers and operations on numbers written in decimal notation which occur in a context of realistic pictures of scenes with a plurality of objects of the same type and with transparent containers of the same type each containing pluralities of objects of the same type. A key insight that is conveyed says that by multiplying the number of containers with the number of objects per container one obtains the total number of objects in a given scene. Assuming that students can count the number of objects in a scene as well as the number of containers in a scene and that they can count the number of objects in a container students can learn that repeated addition comes into play and that having a notation at hand for repeated addition is very practical. This notation is written $a \times b$ and it is used for concrete numerical values mostly. DPII makes no use of variables and does not introduce any general arithmetical laws (that is algebraic laws claimed valid for arithmetical algebras).

Several laws that could have been stated in DPII: for instance that multiplying a number with 1 leaves it unchanged, and commutativity of multiplication could have been illustrated with pictures. But there is no need to do so and DPII refrains from it.

4.1 Expressions as a free format modelling tool

The idea of algebraic expressions in general is that these can be used for systems modelling. Objects in algebraic structures are prime examples of such systems but don't exhaust the options by far. Students are invited to contemplate a picture and to write down an expression that models what they see in the picture. Then they may find out that different models serve the same purpose, that is can each be considered models of the same system (with equal right so to say). That justifies introducing a form of equality. $S_1 = S_2$ now is an assertion about two systems, and students are supposed to express in what sense S_1 and S_2 are the same (they may be the same only partially so to speak) if $S_1 = S_2$ is to be asserted about them.

Students are supposed to get the following message about expressions:

- In many cases there is no fixed meaning for expressions, it is plausible that you determine the meaning of a clans of expressions yourself.
- Expressions and classes of expressions may be designed in order to produce models of realistic situations (embodied in so-called scenes). Invariably such models abstract away most of what makes a scene interesting.
- Using a formal expressions to speak of a model provides a setting from which rigorus analysis may be an option and from which comparison with other (formalised) systems is doable too.

4.2 Static process algebra

Process algebra is an extensive field which deals with mathematical structures for describing sequential behaviors as well as concurrent behaviors. I have spent much time on the design and analysis of process algebras (I may refer to [7], and [5] as highlights of that work). A highly simplified form of process algebra, which I will refer to as static process algebra, ignores sequential behavior and deal with the concurrent existence of static (non-behaving) objects only.

Parallel composition is the principal operation of static process algebra. I will use $- \mid\mid -$ for representing the parallel (concurrent, simultaneous) composition of two systems. I will provide some brief explanation of static process algebra and how it may occur in teaching. One imagines pictures of scenes with apples, pears and oranges. The idea is that a student is asked to grasp a scene in a model, the model being an expression of static process algebra. A, P, and O are constants for the static process algebra at hand.

1. Now $A \mid\mid A$ represents the concurrent presence of two apples. However unusual this notation may seem at first sight, using $- \mid\mid -$ for the concurrent composition of systems has a long tradition in informatics.
2. A scene with two apples, a pear, and an orange may be modeled by $A \mid\mid A \mid\mid P \mid\mid O$, but just as well by $A \mid\mid O \mid\mid P \mid\mid A$. Because both expressions are used as models for the same system they have something in common, that

is that may refer to the same "reality" or be models of the same scene. This is often expressed with an equality sign: $A \mid\mid A \mid\mid P \mid\mid O = A \mid\mid O \mid\mid P \mid\mid A$.

3. When an equality sign is used in between of two expressions, one must always be able to state in words what it is that both expressions are said (and assumed) to have in common. Here both expressions model scenes with the same kinds of entities, each entity occurring the same number of times. The equality asserts that for this kind of modelling of a scene the ordering of occurrence in a parallel composition is immaterial.

4. If $P = Q$ is written as a conclusion or as an assumption then P and Q are considered equal in some sense, a sense which is assumed to be known to the reader, but P and Q need not be the same. Indeed the static process algebra expressions $A \mid\mid A \mid\mid P \mid\mid O$ and $A \mid\mid O \mid\mid P \mid\mid A$ are not the same, but these expressions are equal in the sense just explained.

5. Being liberal about what constitutes a system is quite common in informatics as well. Having some freedom in the design of expressions for systems is nowadays the use rather than the exception. If we denote a box with 2 apples with $b_2(A \mid\mid A)$ (the subscript denoting the maximum capacity of the box) then we find the following system expression for a system S consisting of two of such boxes $b_2(A \mid\mid A) \mid\mid b_2(A \mid\mid A)$. If we introduce $\#_A(-)$ as a function that counts numbers of apples we find $\#_A(b_2(A \mid\mid A)) = \#_A(A \mid\mid A) = 2$ and $\#_A(P \mid\mid Q) = \#_A(P) + \#_A(Q)$ and thus $\#_A(S) = \#_A(b_2(A \mid\mid A) \mid\mid b_3(A \mid\mid A \mid\mid A)) = 2 + 3$.

6. Let $S = b_2(A \mid\mid B) \mid\mid b_2(A \mid\mid B) \mid\mid b_2(A \mid\mid B)$.
Now $\#_A(S) = \#_A(b_2(A \mid\mid B) + \#_A(b_2(A \mid\mid B) + \#_A(b_2(A \mid\mid B)) = 1 + 1 + 1 = 3 \times 1 = 3$.

7. This formalisation of depicted scenes may well precede becoming aware that $2 + 2 = 4, 2 + 3 = 5$ and $3 \times 1 = 1$. It may also add to a more liberal and creative understanding of expressions and equations.

Static process algebra admits further elaboration which I plan to carry out in subsequent work. With the very simple primitives for static process algebra that have just been introduced already a range of useful exercises can be defined. In particular one may ask a student to produce an expression for describing a scene en tho build a scene which is described by a formula.

5 Conclusion

By viewing algebra as a surrounding container of arithmetic it becomes plausible that non-arithmetical algebra precedes arithmetic in a conceivable approach to primary school mathematics. Static process algebra is a form of algebra which may concevably precede arithmetic in teaching. This path of explanation has two virtues: it shows that expressions may be considered models of realistic states of affairs, and it demonstrates that the equality sign is hand made and requires an explanation before it is used. There is never a fixed definition of equality, not even for numbers.

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A Properties of this particular paper

The first Appendix contains information which is specific for this paper, the subsequent Appendices provide the necessary explanation. Frequently a section of paragraph merely contains a poster to the corresponding section of paragraph of a previous document in the MR nopreprint series.

A.1 Licencing

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A.2 Minstroom Research Nopreprint Series Number

This is #8 from the Minstroom Research Nopreprint Series (in brief Minstroom Research NPP#8). The other papers in the series are listed below. In these texts Minstroom Research has been abbreviated to MRbv. I have changed the abbreviation to make it independent of the legal form and to avoid the introduction of an acronym MRbv that is in use for several other meanings already.

1. Minstroom Research NPP#1: "Decision taking avoiding agency", `http://vixra.org/abs/1501.0088` (2015); this paper is not explicitly labeled as a nopreprint but it sufficiently meets the criteria as listed below, though it lacks a defensive novelty analysis which admittedly is a deficiency,

2. Minstroom Research NPP#2: "A nopreprint on algebraic algorithmics: paraconsistency as a afterthought", `http://vixra.org/abs/1501.0203` (2015),

3. Minstroom Research NPP#3: "A nopreprint on the pragmatic logic of fractions", `http://vixra.org/abs/1501.0231` (2015),

4. Minstroom Research NPP#4: "Personal multithreading, account snippet proposals and missing account indications", `http://vixra.org/abs/1502.0204` (2015).
5. Minstroom Research NPP#5: "Terminology for instruction sequencing", `http://vixra.org/abs/1502.0204` (2015).
6. Minstroom Research NPP#6: "A SWOT analysis for Instruction Sequence Theory", `http://vixra.org/abs/1502.0231` (2015).
7. Minstroom Research NPP#7 (I4PSM#1) "Rekenen-informatica": Informatics for Primary School Mathematics, `http://vixra.org/abs/1503.0136` (2015).

A.2.1 NPP Subseries on I4PSM

"Informatics for primary school mathematics (I4PSM)" is coined as the name for a theme within Minstroom Research. For that theme a subseries of the NPP series is maintained. The paper is the second entry in that subseries, which is reflected in its extended code: Minstroom Research NPP#8 I4PSM#2.

A.2.2 Rationale for I4PSM as a theme within Minstroom Research

For this rationale I refer to the corresponding subsection of NPP#7 (I4PSM#1).

A.2.3 Subseries rationale

For this rationale I refer to the corresponding subsection of NPP#7 (I4PSM#1).

A.3 Minstroom Research Document Class

This paper has document class B in the Minstroom Research Document classification scheme. This scheme is detailed in Appendix C. This classification refers to the body of the paper with the exclusion of the Appendices.

A.3.1 Justification of this particular Minstroom Research document classification

In this particular case the classification in class B has the following motivation:

1. The nopreprint status is intentional, submission to a (selectively) peer reviewed publication outlet is not intended. (This indicates Minstroom Research as an appropriate affiliation bringing with the need for classification in A B, C, or D). Forthcoming agreement of any peer review system with the design decisions in the paper is not sought.
2. Subsequent academic research on the basis of the content of this work is not foreseen by the author. Subsequent non-academic research, however, is expected and intended. Working within Minstroom Research towards material that can be used in practice is also intended.

3. The paper contains no novelty claims and does not (intend to) contradict existing literature either.

A.4 Defensive novelty analysis

A nopreprint ought to be equipped with a so-called defensive novelty analysis. An explanation of this this notion as well as an explanation of why it is needed in the case of a nopreprint is given in Appendix B below). For this paper I put forward the following arguments:

- The work only formulates conventions for use of language in further work from MR as well as some directions for further work.
- There is no technical content that might be wrong.
- The paper does not formulate proposals on how other persons ought to work or on how they may understand certain concepts.
- The proposed relation between algebra and arithmetic seems to be inconsistent with the views on this matter by many authors. Yet no claim of novelty can be made given the limited exposure the author had to literature that might analyse the relation between algebra and arithmetic in an educational setting.

B Formalities and policy statements I: about nopreprints

This Appendix begins with brief historical remarks concerning the possibly novel ideas that are put forward in this Appendix as well as and in the following Appendix. The remaining part of this Appendix spells out the details an rational of nopreprints as a novel class of papers and publications.

B.1 Remarks on micro-history

For these remarks see the corresponding section in [2].

B.2 Nopreprints and micro-institutions

This Paragraph and subsequent Paragraphs with are identical (modulo the renaming of MRbv into Minstroom Research) to the corresponding Paragraphs of MR NPP#6, and are not repeated here for that reason.

C Formalities and policy statements 2: using a private micro-institution as an affiliation

This Appendix is identical to Appendix C (modulo the renaming of MRbv into Minstroom Research) of MR NPP#6, it will not be repeated here for that reason.