Graceful Labeling for Trees

Dhananjay P. Mehendale
Sir Parashurambhau College, Pune-411030, India.

Abstract

We establish the existence of graceful labeling for any unlabeled tree by proposing actual construction procedure for such labeling. We define so called lattice and lattice paths sitting inside it. A lattice path is produced by starting with bottom (or top) row of the lattice and choosing one lattice point per row in the lattice in succession and joining these lattice points. With these lattice points we associate vertex pairs representing edges in a complete graph. It obviously follows that each of so called lattice path represents a graceful graph and further it easily follows that there exist in all n! graceful graphs (among which some are trees) in a complete graph on n vertices. In this paper we propose an algorithm to construct graceful labeling for any given unlabeled tree through construction of appropriate lattice path for this tree under consideration.

1. Introduction: A tree on n vertices is said to be graceful or said to have a graceful labeling if when its vertices are labeled with integers {1, 2, …, n} and lines (edges) are labeled by the difference of their respective end vertex labels then all the edge labels taken together constitute the set {1, 2, …, n−1}.

In the year 1964 Ringel [1] proposed the following

Conjecture 1.1(Ringel): If T is a fixed tree with m lines, then $K_{(2m+1)}$, the complete graph on $(2m+1)$ vertices, can be decomposed into $(2m+1)$ copies of T.

Attempts to prove Ringel's conjecture have focused on a stronger conjecture about trees [2], called the Graceful Tree Conjecture:

Conjecture 1.2 (Graceful Tree Conjecture): Every (unlabeled) tree is graceful, i.e. has a graceful labeling.

2. Graceful Tree Conjecture : In this section we show that all graceful trees in a complete graph are essentially certain paths in a triangular shaped lattice of points where with each lattice point we associate a unique vertex pair (edge) as its label.

Definition 2.1: A **delta lattice (n-delta lattice)** is a triangular shaped lattice of points, having shape of an inverted triangle, such that certain distinct vertex pair (edge) has been associated with each point of this lattice, and the vertex pairs (each representing a unique edge) are assigned to these lattice points in such a way that the lattice points in the top row have associated labels $(i, i+1)$, where i goes from 1 to n-1, the lattice points in the second row below it have associated labels $(i, i+2)$, where i goes from 1 to n-2, …., the lattice points in the k-th row, reached by successively creating rows downwards, have associated labels $(i, i+\mathrm{k})$, where i goes from 1 to n-k, … the last row has a single lattice point with vertex pair $(1, n)$ as the associated label.

We give below as an illustration the representation of this lattice with associated labels for n = 2,3,4,5,6 (we don't draw here the associated lattice points and it is to be understood that they are there) as follows:

1) For n = 2, the 2-delta lattice consists of single lattice point labeled by the associated vertex pair (1,2) :

(1,2)

2) For n = 3, the 3-delta lattice is:

(1,2) (2,3)

(1,3)

3) For n = 4, the 4-delta lattice is:

(1,2) (2,3) (3,4)

(1,3) (2,4)

(1,4)

4) For n = 5, the 5-delta lattice is:

(1,2) (2,3) (3,4) (4,5)

(1,3) (2,4) (3,5)

(1,4) (2,5)

(1,5)

5) For n = 6, the 6-delta lattice is:

(1,2) (2,3) (3,4) (4,5) (5,6)

(1,3) (2,4) (3,5) (4,6)

(1,4) (2,5) (3,6)

(1,5) (2,6)

(1,6)

Definition 2.2: An imaginary vertical line starting from lattice point associated with pair (1,*n*) and going upwards passing through the lattice points (2,*n*-1), (3,*n*-2), ….., extending and incorporating the lattice points on the rows, and rising up to first row is called **line of symmetry**.

In the above illustrations of delta lattices:

1) For $n = 2$ the line of symmetry passes through lattice point associated with vertex pair (1, 2), i.e. through the only lattice point.
2) For $n = 3$ the line of symmetry passes through lattice point associated with vertex pair (1, 3), since there is no other lattice point on this vertical line.
3) For $n = 4$ the line of symmetry passes through lattice point associated with vertex pairs (1, 4), (2, 3) since there is no lattice point on this vertical line from second row.
4) For $n = 5$ the line of symmetry passes through lattice point associated with vertex pairs (1, 5), (2, 4) since there is no lattice point on this vertical line from first and third row.

5) For $n = 6$ the line of symmetry passes through lattice point associated with vertex pairs (1, 6), (2, 5), (3, 4) since there is no lattice point on this vertical line from second and fourth row.
6) For $n = 7$ the line of symmetry passes through lattice point associated with vertex pairs (1, 7), (2, 6), (3, 5) since there is no lattice point on this vertical line from first, third, and fifth row.

Definition 2.3: If we choose one entry (lattice point in terms of vertex pair) from each row of the triangular lattice (and consider the graph produced by edges in this choice taken together) then this assembly of vertex pairs taken together in a set is called a **lattice path.** (It is easy to check that in a lattice, corresponding to graph of n vertices, each lattice path among all possible n! lattice paths represents a graceful graph, i.e. a labeled graph having graceful labeling).

Definition 2.4: If a lattice path formed by choosing vertex pairs such that each row of the lattice contributes exactly one vertex pair and all vertex pairs taken together contain all the vertices labeled as {1, 2, 3, ….., n} then each of such lattice paths represent a **graceful tree and these are the only lattice paths representing graceful trees.** All other lattice paths formed by choosing one vertex pair from each row of the triangular lattice but which vertex pairs taken together do not contain all the vertices are not trees though they are graceful graphs.

We now proceed to give examples of the lattice paths in the above mentioned lattices for n = 2, 3, 4, 5, 6

1) Case $n = 2$: In this case, there is only one lattice point with associated vertex label (1, 2). So, the lattice path is of zero length.
2) Case $n = 3$: In this case, there two lattice paths formed by vertex pairs {(1, 3), (1, 2)} and {(1, 3), (2, 3)}
3) Case $n = 4$: In this case, there are in all six (3!) lattice paths formed by vertex pairs {(1, 4), (1, 3), (1, 2)}, {(1, 4), (1, 3), (2, 3)}, {(1, 4), (1, 3), (3, 4)}, {(1, 4), (2, 4), (1, 2)}, {(1, 4), (2, 4), (2, 3)}, {(1, 4), (2, 4), (3, 4)}. Out of these the first, second, fourth, and fifth lattice paths are graceful trees, while paths third and sixth are only graceful graphs but not graceful trees.
4) Case $n = 5$: The three lattice paths which are nonisomorphic graceful trees are

$$\{(1,5), (1,4), (1,3), (1,2)\}$$

{(1,5), (1,4), (1,3), (2,3)}
{(1,5), (1,4), (2,4), (2,3)}

5) Case $n = 6$. In this case we get following seven distinct lattice paths which correspond to nonisomorphic trees and chosen entries in successive rows are shown to be joined by an arrow to bring clarity about path structure:

(1,6)→(1,5)→(1,4)→ (1,3)→(1,2) …..(1)

(1,6)→(1,5)→(1,4)→ (1,3)→ (2,3) ….(2)

(1,6)→(1,5)→(1,4)→ (2,4)→ (2,3) .....(3)

(1,6)→(1,5)→(2,5)→(2,4)→(2,3) .....(4)

(1,6)→(1,5)→(1,4)→(2,4)→(3,4) …..(5)

(1,6)→(1,5)→(2,5)→(2,4)→(3,4) …..(6)

(1,6)→(1,5)→(2,5)→(1,3)→(3,4) …..(7)

Further, it is easy to see that if we take some lattice path and consider the path **formed as mirror image in the line of symmetry** of the chosen path then both these paths represent graceful graphs which are isomorphic. The graceful nature of mirror image is clear as the mirror image is again a lattice path. More clearly, the mirror image of a lattice point with associated vertex pair (i, j) we get the lattice point with associated vertex pair $(n\text{-}i+1, n\text{-}j+1)$ and so by the below given simple theorem 2.1 the result follows.

Theorem 2.1 Every graceful (n, n−1) tree remains graceful under the transformation (mapping) of vertex labels:
$j \rightarrow (n-j+1)$.

Proof: Let i, k be the vertex labels of two adjacent vertices of the tree. Then the edge label for this edge will be $|i-k|$. Now under the mentioned transformation the edge labels
$|i-k| \rightarrow |(n-i+1)-(n-k+1)| = |i-k|$, hence etc.

□

Remark 2.1: It is easy to visualize that n-delta lattice is essentially a representation for complete graph on n vertices where these vertices are labeled by numbers {1, 2, 3, …, n}.

Definition 2.4: An imaginary vertical line starting from lattice point associated with pair (1,n) and going upwards passing through the lattice points with labels (2,n-1), (3,n-2), ….., extending and incorporating the lattice points on the rows, and rising up to first row is called **line of symmetry**.

Definition 2.5: A **lattice path** is a path obtained by selecting some one lattice point on each row of n-delta lattice and joining these lattice points in sequence starting with the lattice point on the lowest row and moving up in succession incorporating the chosen lattice point on each row till the path finally terminates at the selected lattice point on the first row.

Definition 2.5: A **piece** of a lattice path is certain portion of lattice path, may consists of single lattice point with label (i, j) that represents an edge with end vertices having labels i and j, or certain portion of lattice path obtained by joining some lattice points in successive rows of n-delta lattice, or the entire lattice path itself.

Remark 2.2: In the above definition by starting with the selected lattice point on the first row and moving down in succession incorporating the chosen lattice point on each row till the path finally terminates at the selected lattice point on the last row we will construct the same lattice path.

Remark 2.3: It is easy to visualize that a lattice path in n-delta lattice, when corresponds to a tree, is essentially equivalent to showing existence of a gracefully labeled isomorphic copy (for an unlabeled tree of some isomorphic type) in the complete graph on n vertices where these vertices are labeled by numbers {1, 2, 3, …, n}.

It is easy to see that when we take a lattice path and use it to construct a graph by taking the vertex pairs that appear in that path as edges and the numbers that appear in the totality in these vertex pairs as vertex labels we get essentially a graceful graph.

If this graceful graph is an $(n, n\text{-}1)$ connected graph or $(n,n\text{-}1)$ acyclic graph then the lattice path represents a graceful tree. Otherwise, the associated graph, though graceful, obtained from that lattice path is not a tree.

Consider following two **straight** lattice paths which are symmetrically placed (mirror images of each other) around line of symmetry, namely,

$$(1,n) \rightarrow (1,n-1) \rightarrow (1,n-2) \rightarrow \ldots\ldots\ldots \rightarrow (1,2)$$

and

$$(1,n) \rightarrow (2,n) \rightarrow (3,n) \rightarrow \ldots\ldots\ldots\ldots \rightarrow (n-1,n)$$

It is easy to check that these lattice paths lying at left and right boundary of n-delta lattice correspond as a graph to gracefully labeled (n,n-1) **star trees**.

Consider following two **zigzag** lattice paths which are symmetrically placed (mirror images of each other) around line of symmetry, namely,

$$(1,n) \rightarrow (1,n-1) \rightarrow (2,n-1) \rightarrow (2,n-2) \rightarrow (3,n-2) \rightarrow \ldots\ldots$$

and

$$(1,n) \rightarrow (2,n) \rightarrow (2,n-1) \rightarrow (3,n-1) \rightarrow (3,n-2) \rightarrow (4,n-2) \rightarrow \ldots$$

It is easy to check that these lattice paths going away from and coming towards line of symmetry by unit distance at each alternate move and passing in a zigzag way close to line of symmetry of n-delta lattice correspond as a graph to gracefully labeled (n,n-1) **path-trees**.

We now proceed with an algorithm to generate all possible gracefully labeled trees in terms of the totality of all lattice paths in n-delta lattice.

For the sake of clarity let us state some more definitions:

Definition 2.8: A tree is called a **star-tree** or simply a **star** if it is a tree with one vertex of degree k, k bigger than one, and all other vertices are adjacent to it and have degree exactly equal to one.

Definition 2.9: A tree is called a **path-tree** or simply a **path** if it is tree with all vertices have degree two except two (end) vertices (where the path terminates) and they have degree one.

3. Graceful Labeling of Tress by Graceful Labeling of Paths Composing the Tree: We now proceed with most transparent **constructive proof for Graceful Tree Conjecture** using the **important idea** of representing any desired tree among the arbitrary (n, n-1)-trees in terms of the **juxtaposition of paths joined in a predefined way** and the by gracefully labeling individual paths by choosing them in the n-delta lattice in the required way to match the starting and ending points of these paths in the n-delta lattice to give rise to graceful avatar of desired tree!

Theorem 3.1: Every tree can be looked upon as made up of juxtaposition of **paths (or path-trees)** of certain lengths in a predefined way so that when they are selected in the n-delta lattice with proper care of matching the starting and ending points of these paths then they together give rise to the desired graceful tree.

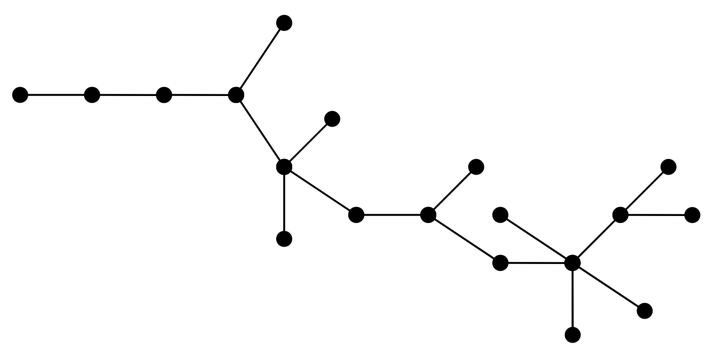
Proof: The proof will follow in a constructive way from following the steps of the **Construction Algorithm** given below:

1) Consider the given unlabeled (n, n-1)-tree for successive labeling of vertices.
2) Starting with a pendant vertex go along some path terminating in some other pendant vertex of length L_1 say.
3) Construct n-delta lattice. Starting with vertex pair (1, n) at the bottom of this lattice, proceed to build a path of length L_1 by moving up words in zigzag fashion.
4) Construct this part of the total lattice path to be built by joining in succession the chosen vertex pairs and also label those vertices of the unlabeled (n, n-1)-tree to show this path.
5) Find the vertex nearest to starting pendant vertex of the labeled path in step 4) from where a new path emerges. Let that path starting from some vertex of earlier path and terminating in some pendant vertex be of length L_2.
6) Starting at this already labeled vertex in the vertex pair on the constructed lattice path proceed in a zigzag way to construct this part of the lattice path incorporating some new vertex pairs and using the labels in the newly chosen vertex pairs also label those vertices on this path of length L_2 emerging from some labeled vertex on first path and terminating in some pendant vertex of the given unlabeled (n, n-1)-tree.

7) Continue to construct in this way firstly the further part of the lattice path and secondly the labeling of the vertices incorporated in further part of the original unlabeled tree. Repeating this step we complete the construction of the lattice path corresponding to given unlabeled tree and also gracefully labeled avatar of this unlabeled tree.

□

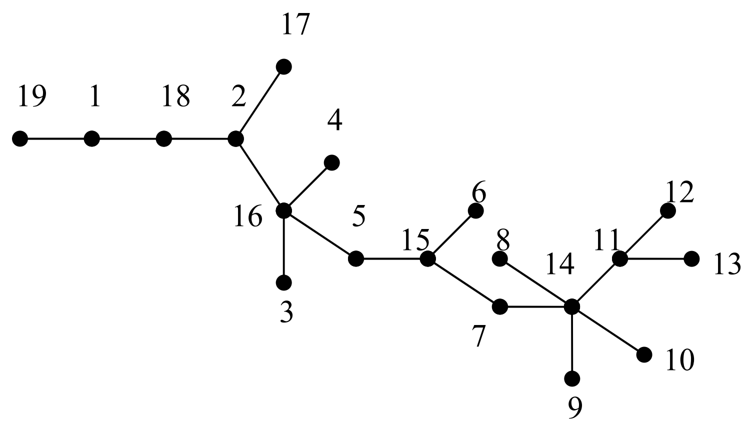
Example 3.1: Consider following unlabelled (19, 18)-tree of a tree



In the **19-delta lattice** we consider this tree as made up of following path-trees juxtaposed in a predefined way and are written after the path-trees their corresponding parts of the lattice path.

1) Starting path-tree: 19→1→18→2→17. Part of lattice path representing this path tree: {(1,19), (1,18), (2,18), (2,17)}
2) Next path-tree starting at vertex labeled 2: 2→16→3. Part of lattice path representing this path tree: {(2,16), (3,16)}
3) Next path-tree starting at vertex labeled 16: 16→4. Part of lattice path representing this path tree: {(4,16)}
4) Next path-tree starting at vertex labeled 16: 16→5→15→6. Part of lattice path representing this path tree: {(5,16), (5,15), (6,15)}
5) Next path-tree starting at vertex labeled 15: 15→7→14→11→12. Part of lattice path representing this path tree: {(7,15), (7, 14), (11,14), (11,12)}
6) Next path-tree starting at vertex labeled 8: 8→14→10. Part of lattice path representing this path tree: {(8,14), (10,14)}
7) Next path-tree starting at vertex labeled: 14→9. Part of lattice path representing this path tree: {(9,14)}
8) Next Path starting at vertex labeled: 11→13. Part of lattice path representing this path tree: {(11,13)}

This choosing of paths in succession leads to the following gracefully labeled avatar of this tree!



The (complete) lattice path in the 19-delta lattice, made up of the above parts of lattice path corresponding to path-trees chosen to cover given tree is

(1,19)→(1,18)→(2,18)→(2,17)→(2,16)→(3,16)→(4,16)→(5,16)→(5,15)→(6,15)→(7,15)→(7,14)→(8,14)→(9,14)→(10,14)→(11,14)→(11,13)→(11,12)

Note that choosing of path-trees or paths is completely arbitrary and one can choose any other paths instead of these paths. The only essential requirement while choosing paths is that **they all together when properly juxtaposed should produce the tree isomorphic to the original tree** for which this representation in terms of paths is produced, i.e. these path-trees should form a **proper cover** (incorporating all vertices and edges) for the tree under consideration and **should produce lattice path corresponding to the graceful avatar for given unlabeled** (and labeled subsequently through successive labeling of individual chosen path-trees covering this unlabeled tree) **tree in the given n-delta lattice**.

References

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2. West Douglas B., Introduction to Graph Theory, Pages 69-73, Prentice-Hall of India, Private Limited, New Delhi 110-001, 1999.